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Crystallographic orbifolds: towards a classification of unitary conformal field theories with central charge $c = 2$

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Abstract: We study the moduli space $\mathcal{C}^2$ of unitary two-dimensional conformal field theories with central charge $c = 2$. We construct all the 28 nonexceptional non-isolated irreducible components of $\mathcal{C}^2$ that may be obtained by an orbifold procedure from toroidal theories. The parameter spaces and partition functions are calculated explicitly, and all multicritical points and lines are determined. We show that all but four of the 28 irreducible components of $\mathcal{C}^2$ corresponding to nonexceptional orbifolds are directly or indirectly connected to the moduli space of toroidal theories in $\mathcal{C}^2$. We relate our results to those by Dixon, Ginsparg, Harvey on the classification of $c = 3/2$ superconformal field theories and thereby give geometric interpretations to all nonisolated orbifolds discussed there.

Keywords: Conformal and W Symmetry, Discrete and Finite Symmetries.

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Contents

1. Introduction

In this paper we study the moduli space $\mathcal{C}^2$ of unitary two-dimensional conformal field theories with central charge $c = 2$. The component $\mathcal{T}^2$ of the moduli space corresponding to compactification on a two-dimensional torus is well understood [3, 17]. One can conjecture that every theory in $\mathcal{C}^2$ either corresponds to compactification on a torus or on an orbifold thereof. It was stated in [4] that it is not difficult to classify all possible types of $c = 2$ (symmetric) orbifold models which can be obtained by modding out an automorphism group of a theory in $\mathcal{T}^2$. However, to our knowledge this analysis has not been carried out explicitly up to now.

The paper is organized as follows: in section 2 we briefly review the features of $\mathcal{T}^2$ relevant to our studies. Moreover, we argue that apart from some exceptional cases any nonisolated component of $\mathcal{C}^2$ which can be constructed by applying an orbifold procedure to a subspace of the Teichmüller space of $\mathcal{T}^2$ can be obtained by modding out an automorphism group of a two-dimensional torus. This means that to find all such nonisolated components we can use the standard classification of crystallographic groups in two dimensions, which is discussed in section 3. Section 4 contains a case by case study of all the 28 irreducible components of $\mathcal{C}^2$ obtained from $\mathcal{T}^2$ by modding out crystallographic groups. All consistent choices of the B-field on the original toroidal theory and the effect of discrete torsion are discussed, which also leads to some insight into the role of the B-field in a conformal field theory. We explicitly calculate the corresponding partition functions and determine the parameter space for each component. In section 5 we make use of results of B. Rostand's [20, 21] to determine all intersections of the irreducible components of the moduli space obtained in section 4, i.e. singular or multicritical lines and points in $\mathcal{C}^2$. This also sheds some light on the effect of discrete torsion. We find a whole wealth of fourteen multicritical lines and 31 multicritical points for the crystallographic components, among them three quadrucritical and ten tricritical points. In particular, we show that all but four of the components of $\mathcal{C}^2$ constructed by crystallographic orbifolds are directly or indirectly connected to $\mathcal{T}^2$. The moduli space exhibits a complicated graph like structure with many loops. In section 6 we discuss theories obtained as tensor products of known models with central charge $c < 2$. We relate our results to those on $c = 3/2$ superconformal field theories [6] and are able to interprete all the orbifolds discussed there in terms of crystallographic orbifolds.

Unitary two-dimensional quantum field theories can be described as minkowskian theories on the circle or equivalently as euclidean theories on the torus with parameter σ in the upper half plane. The world sheet coordinates are called ξ_0, ξ_1, and we frequently use $z = e^{\xi_0 + \sigma\xi_1}$, $z \in Z$ to parametrize the worldsheet on an annulus $Z \subset \mathbb{C}^*$.

2. The moduli space $\mathcal{T}^2$ of toroidal theories

Let us briefly recall the structure of the moduli space $\mathcal{T}^2$ of theories corresponding to toroidal compactification in two dimensions (see also [4]). Consider a torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$, where $\Lambda \subset \mathbb{R}^2$ is a nondegenerate lattice with generators $\lambda_1, \lambda_2 \in \Lambda$. The nonlinear σ-model on $\mathbb{T}^2$ describes two real massless scalar fields $\Phi^\mu : Z \to \mathbb{T}^2$, $\mu \in \{1,2\}$, governed by the action

$$S = \frac{1}{2\pi} \int_Z d^2z \, (G_{\mu\nu} + B_{\mu\nu}) \partial\Phi^\mu(z,\bar z) \bar\partial \Phi^\nu(z,\bar z) \,, \tag{2.1}$$

where we have set $\alpha' = 1$ by choosing a unit of length. The constant symmetric tensor $G_{\mu\nu} = \langle \lambda_\mu, \lambda_\nu \rangle$ defines the metric on $\mathbb{T}^2$ and the antisymmetric tensor $B_{\mu\nu} = -B_{\nu\mu}$ is known as B-field. In other words, by a slight abuse of notation the parameters of the theory are

$$(\Lambda, B) \in \mathrm{O}(2)\backslash \, \mathrm{Gl}(2) \times \mathrm{Skew}(2) \,. \tag{2.2}$$

Each Φ^μ in (2.1) decomposes into a left- and a rightmoving part $\Phi^\mu(z,\bar z) = \frac{1}{2}(\varphi^\mu(z) + \bar\varphi^\mu(\bar z))$, $\mu \in \{1,2\}$. The fields $j_\mu = i\partial\varphi^\mu$ are the two generic abelian u(1) currents of the theory which generate translations along the coordinate axes of $\mathbb{T}^2$. The energy momentum tensor is given in the Sugawara form

$$T = \frac{1}{2} \left(:j_1 j_1: + :j_2 j_2: \right) , \quad \bar T = \frac{1}{2} \left(:\bar j_1 \bar j_1: + :\bar j_2 \bar j_2: \right) . \tag{2.3}$$

In the following, we will work with φ^μ and $\bar\varphi^\mu$ separately, but the left-right transformed analogue of some statement will often not be mentioned explicitly in order to avoid tedious repetitions.

The Hilbert space $\mathcal{H}$ of our theory decomposes into an infinite number of sectors according to different winding and momentum numbers of the ground state. We label ground states with winding mode $\lambda = m_2\lambda_1 + m_1\lambda_2 \in \Lambda$ and momentum mode $\mu = n_2\mu_1 + n_1\mu_2 \in \Lambda^*$ by $|m_1, m_2, n_1, n_2\rangle$, where (μ_1, μ_2) is the basis dual to (λ_1, λ_2). With

$$(p(\lambda,\mu), \bar p(\lambda,\mu)) := \frac{1}{\sqrt{2}} \left(\mu - B\lambda + \lambda, \mu - B\lambda - \lambda \right) \tag{2.4}$$

and cocycle factors $c_{\lambda,\mu}$ the vertex operator corresponding to $|m_1, m_2, n_1, n_2\rangle$ is

$$V_{\lambda,\mu} := c_{\lambda,\mu} : \exp[ip(\lambda,\mu)\cdot\varphi(z) + i\bar p(\lambda,\mu)\cdot\bar\varphi(\bar z)] : \,. \tag{2.5}$$

In particular, $V_{\lambda,\mu}$ has charge $(p(\lambda,\mu), \bar p(\lambda,\mu))$ with respect to $(j_1, j_2, \bar j_1, \bar j_2)$, and by (2.3) the action of the zero modes $\overset{(-)}{L}_0$ of the Virasoro generators is

$$\overset{(-)}{L}_0 |m_1, m_2, n_1, n_2\rangle = \frac{1}{2} \left(\overset{(-)}{p}(\lambda,\mu) \right)^2 |m_1, m_2, n_1, n_2\rangle \,.$$

Hence our theory has partition function

$$Z_{\Lambda,B}(\sigma) = tr_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = \frac{1}{\eta^2 \bar{\eta}^2} \sum_{\lambda \in \Lambda, \mu \in \Lambda^*} q^{\frac{1}{2}(p(\lambda,\mu))^2} \bar{q}^{\frac{1}{2}(\bar{p}(\lambda,\mu))^2}\,, \tag{2.6}$$

where $q = e^{2\pi i \sigma}$ and $\eta = \eta(\sigma)$ is the Dedekind eta function

$$\eta(\sigma) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,.$$

By [3, 17] toroidal theories are determined uniquely by their charge lattice

$$\Gamma = \Gamma(\Lambda, B) := \{(p(\lambda,\mu), \bar{p}(\lambda,\mu)) \mid (\lambda,\mu) \in \Lambda \oplus \Lambda^*\}\,. \tag{2.7}$$

This is an even unimodular lattice in $\mathbb{R}^{2,2} = \mathbb{R}^2 \times \mathbb{R}^2$ which is equipped with the scalar product

$$(p, \bar{p}) \cdot (p', \bar{p}') := p \cdot p' - \bar{p} \cdot \bar{p}'\,. \tag{2.8}$$

The parameters $(\Lambda, B) \in \mathrm{O}(2) \backslash \mathrm{Gl}(2) \times \mathrm{Skew}(2)$ thus are mapped to $\Gamma(\Lambda, B) \in \mathrm{O}(2) \times \mathrm{O}(2) \backslash \mathrm{O}(2,2;\mathbb{R})$. The moduli space of toroidal conformal field theories with central charge $c = 2$ is

$$\mathcal{T}^2 = \mathrm{O}(2) \times \mathrm{O}(2) \backslash \mathrm{O}(2,2;\mathbb{R}) / \mathrm{O}(2,2;\mathbb{Z}) \tag{2.9}$$

[3, 17]. In the two-dimensional case it is convenient to group the four real parameters $G_{\mu\nu}, B_{\mu\nu}$ of the theory into two complex parameters by

$$\tau = \tau_1 + i\tau_2 := \frac{G_{12}}{G_{22}} + i\frac{\sqrt{\det(G_{\mu\nu})}}{G_{22}}, \quad \rho = \rho_1 + i\rho_2 := B_{12} + i\sqrt{\det(G_{\mu\nu})}. \tag{2.10}$$

Here τ is the image of $\Lambda \in \mathrm{Gl}(2)$ under the natural projection $\mathrm{Gl}(2) \to \mathrm{Sl}(2)^{(\tau)} \cong \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. If $\mathrm{O}(2,2;\mathbb{R}) \ni \Gamma(\Lambda, B) \mapsto (\tau, \rho)$, then $\rho \in \mathbb{H} \cong \mathrm{Sl}(2)^{(\rho)}$, where $\mathrm{Sl}(2)^{(\rho)}$ is the commutant of $\mathrm{Sl}(2)^{(\tau)}$ in $\mathrm{O}(2,2;\mathbb{R})$. Note that τ is the quotient $\int_B dz / \int_A dz$ of the two torus periods (A, B form a symplectic basis of $H_1(\mathbb{T}^2, \mathbb{Z})$) and therefore represents the complex structure of $\mathbb{T}^2$. ρ_2 is the volume of $\mathbb{T}^2$ and specifies the Kähler class, because $\dim_{\mathbb{R}} H^2(\mathbb{T}^2, \mathbb{R}) = 1$ and every metric on a two-dimensional torus is Kähler. Therefore $\rho \in \mathbb{H}$ is called complexified Kähler parameter. Now the generators $\lambda_1, \lambda_2 \in \Lambda$ are given by

$$\lambda_1 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \qquad \text{and} \qquad B = \frac{\rho_1}{\rho_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

By (2.10) for $\lambda = m_2\lambda_1 + m_1\lambda_2 \in \Lambda$ and $\mu = n_2\mu_1 + n_1\mu_2 \in \Lambda^*$ as above (2.4) reads

$$\overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}} \left\{ \begin{pmatrix} n_2\tau_2 \\ -n_2\tau_1 + n_1 \end{pmatrix} + \rho_1 \begin{pmatrix} m_1\tau_2 \\ -m_2 - m_1\tau_1 \end{pmatrix} \underset{(-)}{+} \rho_2 \begin{pmatrix} m_2 + m_1\tau_1 \\ m_1\tau_2 \end{pmatrix} \right\}. \tag{2.12}$$

If (Λ, B) are related to (τ, ρ) by (2.10), for the partition function (2.6) we write

$$Z(\tau_1, \tau_2, \rho_1, \rho_2) := Z_{\Lambda,B}(\sigma) = \frac{1}{\eta^2\bar{\eta}^2} \sum_{\lambda\in\Lambda, \mu\in\Lambda^*} q^{\frac{1}{2}(p(\lambda,\mu))^2} \bar{q}^{\frac{1}{2}(\bar{p}(\lambda,\mu))^2} . \tag{2.13}$$

Note that if $\tau_1 = \rho_1 = 0$, then the torus theory is a tensor product of two theories with $c = 1$ corresponding to compactification of single real bosons on circles of radii $r = \sqrt{G_{22}} = \sqrt{\rho_2/\tau_2}$ and $r' = \sqrt{G_{11}} = \sqrt{\rho_2\tau_2}$. The partition function (2.13) factorizes correspondingly:

$$Z^{c=1}(r) := \frac{1}{|\eta|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{4}(n/r+mr)^2} \bar{q}^{\frac{1}{4}(n/r-mr)^2} , \tag{2.14}$$

$$Z(0, \tau_2, 0, \rho_2) = Z^{c=1}\left(\sqrt{\frac{\rho_2}{\tau_2}}\right) Z^{c=1}(\sqrt{\rho_2\tau_2}) . \tag{2.15}$$

In terms of the new parameters (τ, ρ) the duality group $\mathrm{O}(2, 2; \mathbb{Z})$ in (2.9) translates into the group generated by $\mathrm{PSL}(2, \mathbb{Z}) \times \mathrm{PSL}(2, \mathbb{Z})$, which acts by Möbius transformations on each factor of $\mathbb{H} \times \mathbb{H}$, and the dualities

$$U, V : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}, \qquad U(\tau, \rho) := (\rho, \tau), \qquad V(\tau, \rho) := (-\bar{\tau}, -\bar{\rho}) . \tag{2.16}$$

In terms of the parameters (τ, ρ) the moduli space (2.9) therefore is

$$\mathcal{T}^2 = (\mathbb{H}/\,\mathrm{PSL}(2, \mathbb{Z}) \times \mathbb{H}/\,\mathrm{PSL}(2, \mathbb{Z}))\,/(\mathbb{Z}_2 \times \mathbb{Z}_2). \tag{2.17}$$

By the above interpretation of τ and ρ the duality U interchanges complex and (complexified) Kähler structure of $\mathbb{T}^2$ and is known as mirror symmetry. Compared to the former description (2.9) of the moduli space by equivalence classes of lattices, V correponds to conjugation by $\mathrm{diag}(-1, 1, -1, 1)$ on $\mathrm{O}(2) \times \mathrm{O}(2)\backslash\,\mathrm{O}(2, 2; \mathbb{R})$ which is target space orientation change. Note that world sheet parity which interchanges p and $\bar{p}$ is given by $(\Lambda, B) \mapsto (\Lambda, -B)$ or equivalently $(\tau, \rho) \mapsto (\tau, -\bar{\rho})$ and is not a duality symmetry.

It is not hard to see that the Zamolodchikov metric on $\mathcal{T}^2$ is induced by the product of hyperbolic metrics on each of the factors $\mathbb{H}$ in (2.17). In particular, geodesics on the Teichmüller space $\mathbb{H} \times \mathbb{H}$ of $\mathcal{T}^2$ are well known: The projection on each of the $\mathbb{H}$-factors is a half circle with center on the real axis, a half line parallel to the imaginary axis of $\mathbb{H}$, or constant.

Suppose that a nonisolated component of $\mathcal{C}^2$ with Teichmüller space $\mathcal{E} \subset \mathbb{H} \times \mathbb{H}$ is obtained by modding out a common symmetry group G of all toroidal theories with parameters in $\mathcal{E}$. Assume further that $\mathcal{E}$ is a maximal connected subset of $\mathbb{H} \times \mathbb{H}$ corresponding to theories with symmetry G. In particular, G acts as group of isometries on $\mathcal{E}$, and the $(1, 1)$-fields which describe deformations within $\mathcal{E}$ are invariant under G. Thus $\mathcal{E}$ is totally geodesic.

Let us determine all possible actions of symmetry groups G on theories with parameters in $\mathcal{E}$. Those best understood are of course the ones with *geometric interpretation*, i.e. those induced by an action of G on the torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ of a geometric interpretation (Λ, B). We remark that if $\mathcal{E}$ contains a *large volume theory*, then the action of G does have a geometric interpretation. Namely, in [16, (1.16)] a precise notion of large volume theories was introduced, characterizing them by the fact that for the subset

$$\widetilde{\Gamma} := \Big\{ (p,\bar{p}) \in \Gamma \mid ||p||^2 \ll 1, ||\bar{p}||^2 \ll 1 \Big\} \tag{2.18}$$

of the charge lattice Γ the rank of $\mathrm{span}_{\mathbb{Z}}\widetilde{\Gamma}$ is two. In particular, a large volume theory has a unique preferred geometric interpretation (Λ, B) with large $\rho_2 = \det(G_{\mu\nu})$ in terms of a nonlinear σ model. If $\mathcal{E}$ contains such a large volume theory with preferred geometric interpretation (Λ, B), then the action of G on that toroidal theory will not change this preferred geometric interpretation. Since $\widetilde{\Gamma}$ as defined in (2.18) has the property $\mathrm{span}_{\mathbb{Z}}\widetilde{\Gamma} = \{\frac{1}{\sqrt{2}}(\mu,\mu) \mid \mu \in \Lambda^*\}$, the action of G is given by a geometric symmetry on the corresponding torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$.

Let us assume that G maps the set $\{j_k\bar{\jmath}_l \mid k,l \in \{1,2\}\}$ of generic $(1,1)$ fields of theories in $\mathcal{T}^2$ into itself. By construction of toroidal conformal field theories, this means that G induces an action on the entire Teichmüller space $\mathbb{H}\times\mathbb{H}$ of $\mathcal{T}^2$, which identifies isomorphic theories and fixes $\mathcal{E}$. This action will be denoted $\mathcal{G}$ in the following. By construction (2.17) of the moduli space $\mathcal{T}^2$ of toroidal theories, we must have $\mathcal{G} \subset \mathrm{PSL}(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$. Note that in general $\mathcal{G}$ will be different from G, since an action of G on vertex operators (2.5) by multiplication with phases will be invisible in its induced action on $\mathcal{T}^2$. Moreover, target space orientation change V induces a trivial action on toroidal conformal field theories.

If $\mathcal{E}$ contains a geodesic with the property that its projection on one of the factors of $\mathbb{H}$ in the Teichmüller space is constant, then by (2.12) one checks that $\mathcal{E}$ contains a large volume limit and thus G acts geometrically by the above. Otherwise, since $\mathcal{E}$ is the fixed point set of a subgroup $\mathcal{G} \subset \mathrm{PSL}(2,\mathbb{Z})^2 \times \mathbb{Z}_2^2$, and the action of V need not be discussed, $\mathcal{E}$ must be one of the following spaces (or a Möbius transform thereof):

$$\mathcal{E}_U := \big\{ (\tau_1, t, \tau_1, t) \mid t \in \mathbb{R}^+ \big\}, \qquad \mathcal{E}_{UV} := \big\{ (\tau_1, t, -\tau_1, t) \mid t \in \mathbb{R}^+ \big\}. \tag{2.19}$$

We now argue that in neither of these cases we find new components of $\mathcal{C}^2$ by modding out a nongeometric symmetry. Firstly, since $\mathcal{E}$ is maximal, we may assume $G = \{1, g\}$, where $g \in \{U, UV\}$ and $\mathcal{E} = \mathcal{E}_g$. Then, for mirror symmetry $g = U$ we read off an induced action $n_2 \leftrightarrow m_2$ on the charge lattice (2.12). Moreover, all theories in $\mathcal{E}_U$ have a righthanded $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry, two of whose commuting generators are invariant under this action. Since one checks that all the generic abelian lefthanded U(1) currents are invariant under the action of U as well, we find that the theory we produce by modding out U contains at least two left- and two righthanded abelian currents and thus is a torus theory again. U therefore is SU(2) conjugate to a shift on the charge lattice, which acts by multiplication with $i^{n_2-m_2}$ on states created from the Hilbert space ground state $|m_1, m_2, n_1, n_2\rangle$. It is now a straightforward calculation to check that performing this shift orbifold reproduces the original theory. The case $g = UV$ is treated analogously, since $\mathcal{E}_{UV}$ is obtained from $\mathcal{E}_U$ by a parity change $(\tau, \rho) \mapsto (\tau, -\bar{\rho})$.

Summarizing, up to now we have shown that if the set $\{j_k \bar{j}_l \mid k, l \in \{1, 2\}\}$ of generic $(1, 1)$ fields of theories with parameters in $\mathcal{E}$ is mapped onto itself by the action of G, then this action possesses a geometric interpretation. Otherwise we call the action as well as the corresponding orbifold component of $\mathcal{C}^2$ *exceptional*. In fact, since the Teichmüller space $\mathcal{E}$ of an exceptional component is totally geodesic, to give an estimate of how many exceptional components one may find it suffices to determine all geodesics in $\mathbb{H} \times \mathbb{H}$ that parametrize theories which generically possess more than four $(1, 1)$ fields. By explicit calculation using (2.12) one checks that all such geodesics have the form $f(t) = (\tau_1, t, \pm\tau_1, t) \in \mathbb{H} \times \mathbb{H}, t \in \mathbb{R}^+$, or are Möbius transforms thereof. In other words, without loss of generality $\mathcal{E} = \mathcal{E}_U$ or $\mathcal{E} = \mathcal{E}_{UV}$ as defined in (2.19). Thus in all exceptional cases the toroidal conformal field theories with parameters in $\mathcal{E}$ possess an additional left- or righthanded SU(2) symmetry, and the exceptional action is given by a binary tetrahedral, octahedral or icosahedral subgroup T, O, I of SU(2) (see [14]), possibly in combination with some other symmetry. For instance, if $\tau_1 = 0$ the toroidal theories in $\mathcal{E}_U = \mathcal{E}_{UV}$ decompose into tensor products of $c = 1$ circle theories at radii $r = 1$, $r' = t$, respectively (2.15). Then the possible actions of T, O, I on the first factor theory are clear from the results on conformal field theories with central charge $c = 1$ [14]. In general, exceptional components of $\mathcal{C}^2$ are an interesting issue to be studied separately, which exceeds the scope of the present paper.

We rather concentrate on the nonexceptional components of $\mathcal{C}^2$ in the following. Note that equivalent toroidal theories need not always be mapped onto equivalent orbifold theories if we mod out a symmetry group G, since the action of G in some cases does depend on the particular choice of coordinates on $\mathbb{T}^2$. In other words, $\mathcal{C}$ is obtained from $\mathcal{E}$ by modding out a subgroup of $\{A \in \mathrm{PSL}(2, \mathbb{Z})^2 \rtimes \mathbb{Z}^2 \mid A\mathcal{E} = \mathcal{E}\}$ which needs to be determined for every group G separately.

Recall on the other hand that every theory that was constructed as orbifold by a solvable group G possesses a symmetry which one can mod out to regain the original theory [13, section 8.5]. In section 3 we will see that indeed only orbifolds by solvable groups are of relevance to us. Thus no information distinguishing two theories may be lost under our orbifold procedures. In other words, if we mod out two distinct toroidal theories by the same symmetry, then the resulting theories must be distinct as well.

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3. Symmetries of the two-dimensional torus

By the discussion in section 2, to find the nonexceptional nonisolated orbifold components of the moduli space $\mathcal{C}^2$ we must employ the orbifold procedure for all possible discrete symmetry groups of the torus. In two dimensions, there are seventeen inequivalent crystallographic space groups [19], i.e. discrete subgroups $G \subset \mathrm{O}(2) \ltimes \mathbb{R}^2$ that leave invariant some lattice Λ' and therefore act on a torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$, where $\Lambda \subset \Lambda'$. Figure 1 shows all these symmetry groups by depicting the orbit of some symbol ▶ under G. Each lattice Λ' in figure 1 is formed by fixed combinations of the symbol ▶, which we call motive, in various orientations. Then $\Lambda \subset \Lambda'$ is given by those motives which have the same orientation. The space group G is a semi-direct product of a finite point group $P \subset \mathrm{O}(2)$ and a "translationary" group $\triangle \subset \mathrm{O}(2) \ltimes \mathbb{R}^2$ of elements which do not fix the origin. In figure 1 the group Δ is the minimal subgroup of G which acts transitively on motives. The finite group P is determined by inspection of the particular motives which comprise the orbit of the symbol ▶ under P each.

By the above, P is an automorphism group of the two-dimensional lattice Λ, and if $(S,\delta) \in \Delta$, then there is some $N \in \mathbb{Z}$ such that $N\delta \in \Lambda$. Therefore if $A \in P$ has order M then $M \in \{2,3,4,6\}$. The values $M = 3$ or $M = 6$ require Λ to be a hexagonal lattice ($\tau = e^{2\pi i/3}$); $M = 4$ requires a square lattice ($\tau = i$). As to symmetry groups of order $M = 2$, $\mathbb{Z}_2$ acts by $x \mapsto -x$ as automorphism on every lattice Λ. Moreover, the reflection symmetry group $\mathbb{Z}_2(R)$ is an automorphism group of lattices with $\tau_1 \in \{0, 1/2\}$, where R acts on the coordinates of $\mathbb{T}^2$ by

$$R = R_1 : (x^1, x^2) \mapsto (x^1, -x^2) \qquad \text{or} \qquad R = R_2 : (x^1, x^2) \mapsto (-x^1, x^2)\,. \tag{3.1}$$

Translations $T_\delta = e^{2\pi i p\cdot\frac{\delta}{\sqrt{2}}}$ by $\delta \in \Lambda$ are the basic symmetries of the torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. The result of modding out any torus by a translation symmetry T_δ, $N\delta \in \Lambda$, $N \in \mathbb{N}$ minimal with this property, gives another torus with lattice generated by Λ and δ. To produce a surface different from the torus (and later on non-toroidal conformal field theories), we must combine the translation with the reflection symmetry which we denote $T_R := Re^{2\pi i p\cdot\frac{\delta}{\sqrt{2}}}$. More precisely, we will need this symmetry only in the case $\tau_1 = 0$ and $N = 2$, and we set

$$\begin{gathered}\delta_1 := \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}1/2\\0\end{pmatrix}, \quad \delta_2 := \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}0\\\tau_2/2\end{pmatrix}, \quad \delta' := \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}1/2\\\tau_2/2\end{pmatrix};\\ \text{for } \mu \in \{1,2\}: \ T_{R_\mu} := R_\mu e^{2\pi i p\cdot\frac{\delta_\mu}{\sqrt{2}}}, \qquad T'_{R_\mu} := R_\mu e^{2\pi i p\cdot\frac{\delta'}{\sqrt{2}}}, \qquad \widehat{T}_{R_2} := R_2 e^{2\pi i p\cdot\frac{\delta_1}{\sqrt{2}}}.\end{gathered} \tag{3.2}$$

The groups of type $\mathbb{Z}_2$ generated by T_R or T'_R are denoted $\mathbb{Z}_2(T_R)$ or $\mathbb{Z}_2(\mathbb{T}'_R)$, respectively, where either $R = R_1$ or $R = R_2$. We denote by $A(\theta) \in \mathbb{Z}_M$ the rotation

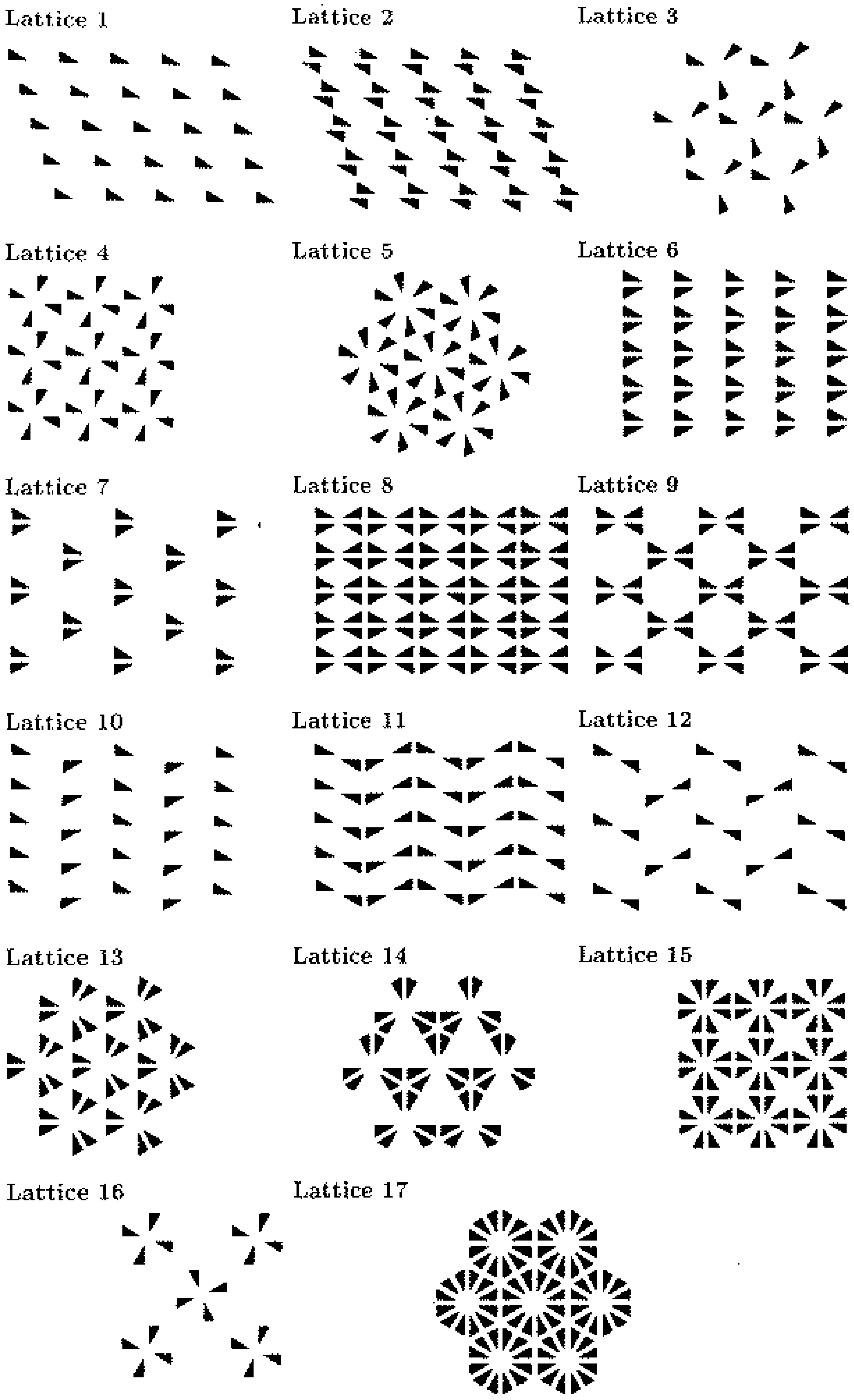

Figure 1: The seventeen inequivalent crystallographic space groups [2].

by an angle of θ. Then $R_2 = A(\pi)R_1$, $\mathbb{T}^{(\prime)}_{R_2} = A(\pi)\mathbb{T}^{(\prime)}_{R_1}$, and we have the noncyclic crystallographic groups

$$
\begin{aligned}
D_2 &:= \{1, A(\pi), R_1, R_2\}\,, & D_3(R) &:= \mathbb{Z}_3 \cup R\mathbb{Z}_3\,,\\
D_4 &:= \mathbb{Z}_4 \cup R_1\mathbb{Z}_4 = \mathbb{Z}_4 \cup R_2\mathbb{Z}_4\,, & D_6 &:= \mathbb{Z}_6 \cup R_1\mathbb{Z}_6 = \mathbb{Z}_6 \cup R_2\mathbb{Z}_6\,,\\
D_2(T_R) &:= \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}\,, & D_2(\mathbb{T}'_R) &:= \{1, A(\pi), \mathbb{T}'_{R_1}, \mathbb{T}'_{R_2}\}\,,\\
D_4(\mathbb{T}'_R) &:= \mathbb{Z}_4 \cup \mathbb{T}'_{R_1}\mathbb{Z}_4 = \mathbb{Z}_4 \cup \mathbb{T}'_{R_2}\mathbb{Z}_4\,.
\end{aligned}
$$

The symmetries that correspond to the lattices in figure 1 are

Lattice	1	2	3	4	5	6	7	8	9
Symmetries	$\{T_\delta, \delta \in \Lambda\}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_2(R)$	$\mathbb{Z}_2(R)$	D_2	D_2

Lattice	10	11	12	13	14	15	16	17
Symmetries	$\mathbb{Z}_2(T_R)$	$D_2(T_R)$	$D_2(\mathbb{T}'_R)$	$D_3(R_1)$	$D_3(R_2)$	D_4	$D_4(\mathbb{T}'_R)$	D_6

(3.3)

Note that as anticipated at the end of section 2, all groups occuring in (3.3) are solvable. This is clear for the abelian groups. For the dihedral groups D_n it follows from the fact that the subgroup $\mathbb{Z}_n$ of rotations in D_n is a normal subgroup with abelian factor $D_n/\mathbb{Z}_n \cong \mathbb{Z}_2$. The finite reflection groups among the groups listed in (3.3) are $\mathbb{Z}_2(R)$, D_2, $D_3(R)$, D_4, and D_6. These are better known as Weyl groups of the semisimple Lie algebras A_1, $A_1 \oplus A_1$, A_2, B_2, and G_2, respectively.

4. Sixteen orbifolds of the torus theories with $c = 2$

In (3.3) we have listed all the seventeen possible symmetry groups G of a two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$. Because the first of them, corresponding to lattice 1, is the translation group $G \cong \Lambda$ which acts trivially on $\mathbb{T}^2$, this implies that we can construct at most sixteen different types of orbifold theories corresponding to different compactifications on $\mathbb{T}^2/G$. To do so, we must show that these symmetries can be continued to symmetries of the corresponding two-dimensional conformal field theories. Since the action of $g \in G$ on the abelian currents j_μ, which generate translations along the coordinate axes of $\mathbb{T}^2$, is determined by the action on $\mathbb{T}^2$, this amounts to continuing every $g \in G$ to a symmetry of the charge lattice (2.7). By (2.12) it is easy to see that this is possible iff $B = g^T B g$. In particular, any of the symmetries listed in (3.3) which corresponds to a lattice characterized by parameters $\tau \in \mathbb{H}$ and $\rho_2 \in \mathbb{R}^+$ immediately gives a symmetry of the toroidal conformal field theory with parameters $(\tau, 0, \rho_2)$, i.e. $B = 0$. But nonzero values for ρ_1 might be possible, too. Note in particular that ρ_1, as parameter in $\mathcal{T}^2$, is only defined modulo $\mathbb{Z}$. In other words, g can be continued to a symmetry of the toroidal conformal field theory iff

$$B = g^T B g + \frac{n}{\rho_2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad n \in \mathbb{Z}\,. \tag{4.1}$$

Below, we will discuss all possible B-field values for each of the symmetry groups listed in (3.3).

Let us recall how we can construct new conformal field theories by modding out a symmetry group G of a conformal field theory with central charge c (see also [5, 7, 8, 9, 12]). First we must project onto group invariant states in the Hilbert space $\mathcal{H}$ of our theory to obtain the untwisted sector of the new theory. In the operator formalism this is achieved by the projection operator $P := \frac{1}{|G|}\sum_{g\in G} g$. We employ the shorthand notation

$$ {}_{g}\underset{1}{\square} := \mathrm{tr}_{\mathcal{H}}\, g q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} $$

to write the untwisted sector partition function as

$$ Z_u = \mathrm{tr}_{\mathcal{H}}\, P q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} = \frac{1}{|G|}\sum_{g\in G} {}_{g}\underset{1}{\square}. \tag{4.2} $$

Z_u is not modular invariant. The reason is that the Hilbert space of the new theory will also contain twisted sectors $\mathcal{H}_f, f \in G$, corresponding to fields which are only well defined on the world sheet of the original theory up to the action of a nontrivial element $f \in G$:

$$ |\varphi\rangle \in \mathcal{H}_f : \qquad \varphi(\xi_0, \xi_1 + 1) = f\varphi(\xi_0, \xi_1)\,. \tag{4.3} $$

More precisely, we should label twisted sectors by conjugacy classes $\{f\}$ of G because φ as in (4.3) also obeys

$$ g\varphi(\xi_0, \xi_1 + 1) = (gfg^{-1})g\varphi(\xi_0, \xi_1) \tag{4.4} $$

for any $g \in G$, and $g\varphi$ is identified with φ, so $\forall\, g \in G : \mathcal{H}_f = \mathcal{H}_{gfg^{-1}}$. $\mathcal{H}^G$ acts by the induced representation on the entire twisted sector.

For $|\varphi\rangle \in \mathcal{H}_f, f \neq 1$ by (4.3) we find that $q_j := \varphi(z = 0)$ is a fixed point of f. If f has J fixed points on $\mathbb{T}^2$ then $\mathcal{H}_f$ decomposes into J isomorphic copies of spaces $\mathcal{H}_f^{(j)}$, $j \in \{1, \ldots, J\}$. If f has order M, then $\varphi^{\pm} := \varphi^1 \pm i\varphi^2$ has mode expansion

$$ \varphi^{\pm}(z) = q_j^{\pm} + i \sum_{n\in\mathbb{Z}\pm 1/M} \frac{1}{n}\alpha_n^{\pm} z^n\,, \tag{4.5} $$

so the corresponding twisted ground state has dimensions

$$ h = \bar{h} = \frac{1}{2}\frac{1}{M}\left(1 - \frac{1}{M}\right) \tag{4.6} $$

(see also [7]). In the twisted Hilbert space $\mathcal{H}_f$, we again have to project onto group invariant states, now by $P_f := \frac{1}{|G|}\sum_{g\in G:\,[g,f]=0} g$. The prefactor is adjusted correctly in order to take care of the multiplicities in each twisted sector. Namely, it takes care of overcounting if later on we sum over all $f \in G, f \neq 1$ instead of conjugacy classes

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$\{f\}$ which actually label twisted sectors by the above (see [7]). We again use the shorthand

$$\underset{f}{{}_{g}\square} := \mathrm{tr}_{\mathcal{H}_f}\, g q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

to write the twisted sector partition function as

$$Z_t = \sum_{f\in G, f\neq 1} \mathrm{tr}_{\mathcal{H}_f}\, P_f q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}} = \frac{1}{|G|} \sum_{\substack{g,f\in G,\\ f\neq 1, [g,f]=0}} \underset{f}{{}_{g}\square}\,. \tag{4.7}$$

The total modular invariant orbifold partition function is

$$Z_{G-\mathrm{orb}} = \sum_{f\in G} \mathrm{tr}_{\mathcal{H}_f}\, P_f q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}} = \frac{1}{|G|} \sum_{\substack{g,f\in G,\\ [g,f]=0}} \underset{f}{{}_{g}\square}\,, \tag{4.8}$$

where we set $\mathcal{H}_1 := \mathcal{H}$ and $P_1 := P$. For general $f, g \in G$ the contribution $\underset{f}{{}_{g}\square}$ can also be calculated by using modular transformations:

$$\underset{f}{{}_{g}\square}\left(-\frac{1}{\sigma}\right) = \underset{g}{{}_{f}\square}(\sigma)\,, \qquad \underset{f}{{}_{g}\square}(\sigma+1) = \underset{f}{{}_{f\circ g}\square}(\sigma)\,. \tag{4.9}$$

Note that $\underset{f}{{}_{g}\square}$ a priori is only defined up to a phase, because the same is true for the action of $g \in G$ on a twisted ground state of $\mathcal{H}_f$. Only if $g = f^k$ for some $k \in \mathbb{Z}$, the phase is fixed by (4.9), and for all other boxes the choice is restricted by modular invariance. For closed modular orbits in the twisted sector there remains an arbitrariness of the phase they contribute with. Here, conjugate subgroups must account with the same phase in order for the representation of G on the twisted sector to be consistent with (4.4). This ambiguity, which by the above does not occur for orbifolds by cyclic groups, is known as discrete torsion [23] and will become relevant in the discussion of lattices 8 and 9 as well as 15–17 below. Because the only groups this will occur in are of type D_2, discrete torsion in these cases will always be given by a choice of sign only.

For nonabelian G, (4.8) can be written as sum over abelian subgroups of G with overcounted terms subtracted off. To do so, we call a subgroup $H \subsetneqq G$ maximal abelian if there is no abelian $G' \subsetneqq G$ such that $H \subsetneqq G'$. We also introduce multiplicities $n_{H'} := \#\{H \subsetneqq G \text{ maximal abelian } | H' \subsetneqq H \text{ maximal}\}$ and find

$$Z_{G-\mathrm{orb}} = \frac{1}{|G|}\left(\sum_{\substack{H\subsetneqq G \text{ max.}\\ \text{abelian}}} |H| Z_{H-\mathrm{orb}} - \sum_{\substack{H'\subseteqq G:\, \exists H\subsetneqq G\\ \text{max. abelian},\, H'\subsetneqq H}} (n_{H'}-1) Z_{H'-\mathrm{orb}} \right). \tag{4.10}$$

4.1 Lattices 2 to 5: $\mathbb{Z}_M$ orbifold theories

We briefly describe the $\mathbb{Z}_M$ orbifold construction. For details see [10], where the $\mathbb{Z}_M$ orbifold partition functions were constructed for $c = 3$ superconformal field theories. Most of the arguments translate directly to the purely bosonic case with $c = 2$ studied here.

In the following, let γ be a generator of $\mathbb{Z}_M$ and assume $\mathbb{T}^2 = \mathbb{R}^2/\Lambda$ to be a torus with $\mathbb{Z}_M$ symmetry, where Λ is characterized by specific values of τ and ρ_2 as explained in section 2. By the discussion in section 3 this means that $\tau = e^{2\pi i/M}$ if $M \in \{3, 4, 6\}$, and $\rho_2 \in \mathbb{R}^+$ arbitrary, whereas $\mathbb{Z}_2$ is a symmetry for every torus. Because by (2.11) γ commutes with B for any value of ρ_1, from (4.1) we know that every toroidal conformal field theory with parameters $(\tau, \rho) \in \mathbb{H} \times \mathbb{H}$, $\tau = e^{2\pi i/M}$ for $M \in \{3, 4, 6\}$, has $\mathbb{Z}_M$ symmetry. The action of the rotation group $\mathbb{Z}_M$ on the charge lattice (2.7) is given by

$$\gamma \in \mathbb{Z}_M\,, \qquad \gamma : (p, \bar{p}) \mapsto (\gamma p, \gamma \bar{p})\,. \tag{4.11}$$

It follows that the $\mathbb{Z}_M$ action commutes with Möbius transformations on ρ. The $\mathbb{Z}_2$ action commutes with the entire $\mathrm{PSL}(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$ of (2.17), so for the families of $\mathbb{Z}_M$ orbifold conformal field theories with $c = 2$ we get the following irreducible components of $\mathcal{C}^2$:

$$\begin{aligned} & \mathcal{C}_{\mathbb{Z}_2-\mathrm{orb}} \cong \mathcal{T}^2\,, \\ \text{for } M \in \{3,4,6\}: \quad & \mathcal{C}_{\mathbb{Z}_M-\mathrm{orb}} = \left\{(\tau,\rho) \mid \tau = e^{2\pi i/M}, \rho \in \mathbb{H}/\,\mathrm{PSL}(2,\mathbb{Z})\right\} \\ & \phantom{\mathcal{C}_{\mathbb{Z}_M-\mathrm{orb}}} \cong \mathbb{H}/\,\mathrm{PSL}(2,\mathbb{Z})\,. \end{aligned} \tag{4.12}$$

By (4.11) the Hilbert space sectors built on the ground states $|m_1, m_2, n_1, n_2\rangle$ are permuted by the $\mathbb{Z}_M$ action, the only fixed ground state being $|0,0,0,0\rangle$. Since the $|m_1, m_2, n_1, n_2\rangle$ are pairwise orthogonal, the only contribution to $\sum_{k=1}^{M-1} \gamma^k \underset{1}{\square}$ in (4.2) comes from the Hilbert space sector built on $|0,0,0,0\rangle$. The $\mathbb{Z}_M$ action on oscillator modes is read off from

$$(\gamma^k \varphi^\pm)(z) = e^{\pm\frac{2\pi i k}{M}} \varphi^\pm(z)\,, \qquad k \in \{1, 2, \ldots, M-1\}\,. \tag{4.13}$$

This allows to construct the untwisted sector partition function (4.2). The twisted sector partition function (4.7) is either obtained by using (4.6) and (4.5) to calculate every box $\gamma^k \underset{\gamma^l}{\square}$, $l \neq 0$, separately or by modular transformations.

Lattice 2: the $\mathbb{Z}_2$ orbifold. By (4.12) lattice 2 depicts an arbitrary lattice. (4.2) and the above show that for any $(\tau, \rho) \in \mathbb{H} \times \mathbb{H}$

$$Z_u = \frac{1}{2}\left(Z(\tau,\rho) + \frac{(q\bar{q})^{-1/12}}{\prod_{n=1}^\infty (1+q^n)^2(1+\bar{q}^n)^2}\right) = \frac{1}{2}\left(Z(\tau,\rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right).$$

Here and in the following $\vartheta_i(y,\sigma)$, $i \in \{1,\ldots,4\}$ denote the classical Jacobi theta functions, and $\vartheta_i(\sigma) := \vartheta_i(0,\sigma)$.

Every torus $\mathbb{T}^2$ has four fixed points under the $\mathbb{Z}_2$ symmetry. By (4.6) this yields four twisted ground states with conformal dimensions $(h, \bar{h}) = (1/8, 1/8)$. By (4.7) we find for the twisted sector partition function

$$\begin{aligned} Z_t &= 4 \cdot \frac{1}{2} (q\bar{q})^{-1/12} \left(\left| q^{1/8} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{-2} \right|^2 + \left| q^{1/8} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^{-2} \right|^2 \right) \\ &= 4 \cdot \frac{1}{2} \left(\left| \frac{\eta(\sigma)}{\vartheta_4(\sigma)} \right|^2 + \left| \frac{\eta(\sigma)}{\vartheta_3(\sigma)} \right|^2 \right). \end{aligned} \tag{4.14}$$

The complete $\mathbb{Z}_2$ orbifold partition function is

$$Z_{\mathbb{Z}_2-\mathrm{orb}}(\tau, \rho) = \frac{1}{2} \left(Z(\tau, \rho) + 4 \left| \frac{\eta(\sigma)}{\vartheta_2(\sigma)} \right|^2 + 4 \left| \frac{\eta(\sigma)}{\vartheta_4(\sigma)} \right|^2 + 4 \left| \frac{\eta(\sigma)}{\vartheta_3(\sigma)} \right|^2 \right). \tag{4.15}$$

The analogous formula for $\mathbb{Z}_2$ orbifold conformal field theories with $c = 1$ is of course also well known [11, 22, 24]:

$$Z^{c=1}_{\mathrm{orb}}(r) = \frac{1}{2} \left(Z^{c=1}(r) + 2 \left| \frac{\eta(\sigma)}{\vartheta_2(\sigma)} \right| + 2 \left| \frac{\eta(\sigma)}{\vartheta_4(\sigma)} \right| + 2 \left| \frac{\eta(\sigma)}{\vartheta_3(\sigma)} \right| \right), \tag{4.16}$$

where $Z^{c=1}(r)$ was given in (2.14).

Lattice 3: the $\mathbb{Z}_3$ orbifold. Lattice 3 has $\tau = e^{2\pi i/3}$ and by (4.12) we may pick arbitrary $\rho \in \mathbb{H}/\operatorname{PSL}(2, \mathbb{Z})$. The untwisted sector partition function is

$$Z_u = \frac{1}{3} \left(Z(\tau = e^{2\pi i/3}, \rho) + 6 \left| \frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3}, \sigma)} \right|^2 \right).$$

$\mathbb{Z}_3$ symmetric tori have three fixed points under the $\mathbb{Z}_3$ action. Thus by (4.6) there are three twisted ground states of dimensions $(h, \bar{h}) = (1/9, 1/9)$ in each of the twisted sectors $\mathcal{H}_\gamma$ and $\mathcal{H}_{\gamma^2}$. The twisted sector partition function therefore is

$$Z_t = 2 \cdot 3 \cdot \frac{1}{3} (q\bar{q})^{-1/18} \sum_{l=1}^{3} \left| \frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{3} + \frac{l}{3}, \sigma)} \right|^2, \tag{4.17}$$

and for the complete $\mathbb{Z}_3$ orbifold partition function,

$$Z_{\mathbb{Z}_3-\mathrm{orb}}(\tau = e^{2\pi i/3}, \rho) = \frac{1}{3} \left(Z + 6 \left| \frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3}, \sigma)} \right|^2 + 6 (q\bar{q})^{-1/18} \sum_{l=1}^{3} \left| \frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{3} + \frac{l}{3}, \sigma)} \right|^2 \right). \tag{4.18}$$

Lattice 4: the $\mathbb{Z}_4$ orbifold. Lattice 4 has $\tau = i$, and by (4.12) we may pick arbitrary $\rho \in \mathbb{H}/\operatorname{PSL}(2,\mathbb{Z})$. The untwisted sector partition function of the $\mathbb{Z}_4$ orbifold can be written as

$$Z_u = \frac{1}{4}\left(Z(\tau = i, \rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{4},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right).$$

Tori with $\tau = i$ have three fixed points under the rotation group $\mathbb{Z}_4$, one of which corresponds to a $\mathbb{Z}_2$ twist and two to $\mathbb{Z}_4$ twists. Hence the total $\mathbb{Z}_4$ orbifold partition function is the sum of untwisted, $\mathbb{Z}_2$, and $\mathbb{Z}_4$ twisted sector partition functions

$$Z_{\mathbb{Z}_4-\mathrm{orb}}(\tau = i, \rho) = Z_u + Z_{2t} + Z_{4t}\,.$$

The $\mathbb{Z}_2$ twisted sector partition function Z_{2t} can be read off from (4.14) by omitting the factor of four. By (4.6), the two ground states in each of the twisted sectors $\mathcal{H}_\gamma$, $\mathcal{H}_{\gamma^2}$ have dimensions $(h,\bar{h}) = (3/32, 3/32)$. The $\mathbb{Z}_4$ twisted sector partition function therefore is

$$Z_{4t} = \frac{1}{4}\left(4\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{4},\sigma)}\right|^2 + 2\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2 + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + 4(q\bar{q})^{-\frac{1}{32}}\sum_{l=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{4}+\frac{l}{4},\sigma)}\right|^2\right).$$

Altogether, we find

$$\begin{aligned} Z_{\mathbb{Z}_4-\mathrm{orb}}(\tau = i, \rho) = \frac{1}{4}\Bigg(& Z(\tau = i, \rho) + 4\sum_{i=2}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{4},\sigma)}\right|^2 + \\ & + 4\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{4},\sigma)}\right|^2 + 4(q\bar{q})^{-1/32}\sum_{l=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{4}+\frac{l}{4},\sigma)}\right|^2\Bigg). \end{aligned} \tag{4.19}$$

Lattice 5: the $\mathbb{Z}_6$ orbifold. Lattice 5 has $\tau = e^{2\pi i/3}$, and by (4.12) we may pick arbitrary $\rho \in \mathbb{H}/(\operatorname{PSL}(2,\mathbb{Z})$. The untwisted sector partition function is

$$Z_u = \frac{1}{6}\left(Z(\tau = e^{2\pi i/3}, \rho) + 2\left|\frac{\eta(\sigma)}{\vartheta_2(\frac{1}{3},\sigma)}\right|^2 + 6\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right).$$

Tori with $\tau = e^{2\pi i/3}$ have three fixed points under the $\mathbb{Z}_6$ rotation symmetry, one corresponding to $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ twists each. The $\mathbb{Z}_6$ orbifold partition function therefore is the sum of untwisted, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ twisted sector partition functions

$$Z_{\mathbb{Z}_6-\mathrm{orb}}(\tau = e^{2\pi i/3}, \rho) = Z_u + Z_{2t} + Z_{3t} + Z_{6t}\,.$$

As before, the $\mathbb{Z}_2$ twisted sector partition function Z_{2t} is obtained from (4.14) by omitting the factor of four. The $\mathbb{Z}_3$ twisted sector partition function Z_{3t} can be read off from (4.17) by omitting the factor of three. By (4.6) the ground states in each of the twisted sectors $\mathcal{H}_\gamma$, $\mathcal{H}_{\gamma^5}$ have dimensions $(h,\bar{h}) = (5/72, 5/72)$, and the $\mathbb{Z}_6$ twisted sector partition function is

$$Z_{6t} = \frac{1}{6}\left(2\left|\frac{\eta(\sigma)}{\vartheta_3(\frac{1}{3},\sigma)}\right|^2 + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{3},\sigma)}\right|^2 + \left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2 + \left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + \right.$$
$$\left. +2(q\bar{q})^{-1/18}\sum_{l=-1}^{1}\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right).$$

Altogether, we obtain

$$\begin{aligned} Z_{\mathbb{Z}_6-\mathrm{orb}}(\tau = e^{2\pi i/3},\rho) = \frac{1}{6}\Bigg(& Z(\tau = e^{2\pi i/3},\rho) + 4\sum_{j=2}^{4}\left|\frac{\eta(\sigma)}{\vartheta_j(\sigma)}\right|^2 + \\ & + 2\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{1}{3},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 + \\ & + (q\bar{q})^{-\frac{1}{18}}\sum_{l=-1}^{1}\left(2\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right)\Bigg). \end{aligned} \tag{4.20}$$

4.2 Lattices 6 to 17: modding out by R or $T_R^{(l)}$ reflection symmetries

The reflection symmetry R is a symmetry for every lattice with $\tau_1 \in \{0, 1/2\}$. By inspection of the action on the respective fundamental cell one easily checks that an exchange of R_1 and R_2 is equivalent to a transformation of τ_2; we define

$$S, T \in \mathrm{PSL}(2,\mathbb{Z}): \qquad S: \zeta \mapsto -\frac{1}{\zeta}, \quad T: \zeta \mapsto \zeta + 1;$$
$$\Theta := \begin{cases} S & \text{if } \zeta_1 = 0, \\ TS\mathbb{T}^2S & \text{if } \zeta_1 = \frac{1}{2}. \end{cases}$$

Then

$$R_1 \leftrightarrow R_2 \text{ is equivalent to } \tau \mapsto \Theta\tau, \text{ where } \begin{cases} \Theta(i\tau_2) = \frac{i}{\tau_2}, \\ \Theta\left(\frac{1}{2} + i\tau_2\right) = \frac{1}{2} + \frac{i}{4\tau_2}. \end{cases} \tag{4.21}$$

We can therefore restrict ourselves to the discussion of the symmetry R_1 in the following. To extend R_1 to the charge lattice (2.7), the B-field B must obey (4.1) which is true iff $\rho_1 \in \frac{1}{2}\mathbb{Z}$. Then by using (2.12) for the R_1 action $|m_1, m_2, n_1, n_2\rangle \mapsto |m_1', m_2', n_1', n_2'\rangle$ we obtain

$$\begin{aligned} m_1' &= -m_1\,, & n_1' &= -n_1 + 2\tau_1 n_2 + 2\rho_1 m_2 + 4\tau_1\rho_1 m_1\,, \\ m_2' &= m_2 + 2\tau_1 m_1\,, & n_2' &= n_2 + 2\rho_1 m_1\,, \end{aligned} \tag{4.22}$$

and the invariant vectors $(p, \bar{p})$ of the charge lattice correspond to $|0, m_2, n_1, n_2\rangle$,

$$\overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} n_2\tau_2 \pm m_2\rho_2 \\ 0 \end{pmatrix}, \quad n_2, m_2 \in \mathbb{Z} \text{ such that } n_1 = n_2\tau_1 + m_2\rho_1 \in \mathbb{Z}\,. \tag{4.23}$$

The Hilbert space ground states $|m_1, m_2, n_1, n_2\rangle$ are pairwise orthogonal, so the only states that give a contribution to ${}_{R_1}\underset{1}{\square}$ are the ones that are built by an action of creation operators on ground states corresponding to vertex operators with R_1-invariant charge vectors (4.23).

Because (4.23) only depends on $\rho_1 \bmod \mathbb{Z}$ the same is true for the resulting orbifold theory and we can pick $\rho_1 \in \{0, 1/2\}$. Note that in the case $\rho_1 = 1/2$ the B-field of our theory is effectively shifted by an integer form if we apply R_1. This will be of some importance below.

To understand the action of the symmetry $T_R^{(\prime)} = RT_{\delta^{(\prime)}}$ on the Hilbert space of a toroidal conformal field theory observe that $T_{\delta^{(\prime)}}$ only acts on the ground state sectors and leaves the oscillator modes invariant. On a state $|m_1, m_2, n_1, n_2\rangle$ corresponding to the charge vector $(p, \bar{p})(\lambda, \mu)$ the action of $T_{R_1}^{(\prime)}$ is given by the action (4.22) of R_1 combined with multiplication by $\exp[2\pi i(p, \bar{p})(\lambda, \mu) \cdot \frac{1}{2}(p, \bar{p})(2\delta^{(\prime)}, 0)] = (-1)^{\langle \mu, 2\delta^{(\prime)}\rangle}$, where we used (2.4). It is therefore a priori clear that as for the action of R we need to restrict the possible B-field values to $\rho_1 \in \{0, 1/2\}$ for consistency of the action of $T_R^{(\prime)}$. By (3.3), $T_R^{(\prime)}$ actions are only needed in the case $\tau_1 = 0$. Using (2.12) one now checks that only for $\rho_1 = 0$ the order of $T_R^{(\prime)}$ is two, whereas for $\rho_1 = 1/2$ we find that $T_R^{(\prime)}$ generates a $\mathbb{Z}_4$ type group. The action of $g := (T_R^{(\prime)})^2$ is given by multiplication with ± 1 on the different Hilbert space sectors. To mod out a toroidal theory A by this $\mathbb{Z}_4$ then is equivalent to performing a $\mathbb{Z}_2$ orbifold procedure on $A/\{1, g\}$. But $A/\{1, g\}$ is another toroidal theory, because both generic torus currents are invariant under g and give conserved currents in $A/\{1, g\}$ as well. The $T_R^{(\prime)}$ action with $\rho_1 = 1/2$ hence need not be considered separately. For $\rho_1 = 0$ by (4.21) we now have

$$T_{R_1}^{(\prime)} \leftrightarrow T_{R_2}^{(\prime)} \text{ is equivalent to } \tau\,(= i\tau_2) \mapsto \Theta\tau \left(= \frac{i}{\tau_2}\right). \tag{4.24}$$

Since by (3.2) $\delta^{(\prime)} = \sqrt{\rho_2/\tau_2}\binom{1/2}{*}$, if $|m_1, m_2, n_1, n_2\rangle$ is R_1-invariant, then by (4.23) $m_1 = 0, n_1 = n_2\tau_1 + m_2\rho_1$, and $T_R^{(\prime)}$ acts by

$$T_R^{(\prime)}: \quad |m_1, m_2, n_1, n_2\rangle \mapsto (-1)^{n_2}|m_1, m_2, n_1, n_2\rangle. \tag{4.25}$$

Below we will construct the families of conformal field theories obtained by the orbifold procedure with a group G which corresponds to one of the lattices 6 to 17. This will yield irreducible components $\mathcal{C}^{(\tau_1,\rho_1)}_{G-\mathrm{orb}}$ of the moduli space $\mathcal{C}^2$ with $\tau_1, \rho_1 \in \{0, 1/2\}$. In some cases discrete torsion gives additional degrees of freedom, increasing the number of irreducible components to $\mathcal{C}^{(\tau_1,\rho_1)}_{G^{\pm}-\mathrm{orb}}$ or even $\mathcal{C}^{(\tau_1,\rho_1)}_{G^{\pm\pm}-\mathrm{orb}}$. The Teichmüller

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space of each such irreducible component is $(\mathbb{R}^+)^k$, where $k = 1$ if τ_2 must be fixed for the particular lattice, too, and $k = 2$ otherwise. To find the correct parameter spaces, we must determine the subgroup $\mathcal{P}$ of $\mathrm{PSL}(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$ in (2.17) which maps the respective Teichmüller space $(\mathbb{R}^+)^k$ onto itself. Then we must discuss which elements of $\mathcal{P}$ map equivalent orbifold theories onto each other.

Restrict $\mathcal{P}$ to one of the factors $\mathbb{R}^+ \subset \mathbb{H}$ of the Teichmüller space $(\mathbb{R}^+)^k$, specified by $\zeta_1 = 0$ or $\zeta_1 = 1/2$. We claim that

$$\mathcal{P} \cap \mathrm{PSL}(2,\mathbb{Z}) = \{1, \Theta\}\,. \tag{4.26}$$

As stated in (4.21), Θ acts on $I^0 := \{\zeta \in \mathbb{H} \mid \zeta_1 = 0\}$ by $\zeta_2 \mapsto 1/\zeta_2$ and on $I^+ := \{\zeta \in \mathbb{H} \mid \zeta_1 = 1/2\}$ by $\zeta_2 \mapsto 1/4\zeta_2$. Now $I^0 = J^0 \cup \Theta J^0$, where $J^0 := \{\zeta \in I^0 \mid \zeta_2 \geq 1\}$. Because J^0 does not contain any two points identified by Möbius transformations, the assertion follows for the case $\zeta_1 = 0$. For $\zeta_1 = 1/2$ observe that $I^+ = (J^+ \cup TSTJ^1) \cup \Theta(J^+ \cup TSTJ^1)$, where $J^+ := \{\zeta \in I^+ \mid \zeta_2 \geq \sqrt{3}/2\}$ and $J^1 := \{\zeta \in \mathbb{H} \mid \|\zeta\| = 1, \zeta_1 \in [-1/2, 0]\}$. Because no two points in $J^+ \cup J^1$ are related by Möbius transformations, the assertion follows. For the respective factor of the Teichmüller space under discussion, Θ will be called T-duality.

By our convention to fix $\tau_1, \rho_1 \in \{0, 1/2\}$ it is clear that target space orientation change $V : (\tau,\rho) \mapsto (-\bar{\tau}, -\bar{\rho})$ in (2.16) can only be contained in $\mathcal{P}$ if $\tau_1 = \rho_1 = 0$, in which case it acts trivially. Mirror symmetry $U : (\tau,\rho) \mapsto (\rho,\tau)$ is contained iff $\tau_1 = \rho_1$ and τ_2 is not fixed. Inspection of the charge lattice (2.7) and the action (4.22) of R_1 shows that mirror symmetry commutes with R_1, R_2 on toroidal conformal field theories. But a priori it is not clear whether it indeed commutes with the action of each of the symmetry groups corresponding to lattices 6 to 17. Therefore, a case by case study is necessary to decide which of Θ, U map a G orbifold onto an equivalent one and thus determine all the parameter spaces $\mathcal{C}^{(\bullet)}_{G^\bullet-\mathrm{orb}}$. We will also see that not all of the lattices yield different components of the moduli space $\mathcal{C}^2$.

4.3 Lattices 6 and 7: the $\mathbb{Z}_2(R)$ reflection orbifold

Lattices 6 ($\tau_1 = 0$) and 7 ($\tau_1 = 1/2$) have reflection symmetry $\mathbb{Z}_2(R)$. For $\tau_1 = \rho_1 = 0$ the torus theory is the product of two $c = 1$ theories corresponding to compactification on a circle each (see (2.15)). The symmetry R_1 by (3.1) leaves the first factor invariant and acts on the second as ordinary $\mathbb{Z}_2$ orbifold. Therefore the resulting partition function is the product of circle and circle orbifold partition function; namely, setting $r := \sqrt{\rho_2/\tau_2}$, $r' := \sqrt{\tau_2\rho_2}$ as in (2.15),

$$Z_{R_1-\mathrm{orb}}(0,\tau_2,0,\rho_2) = Z^{c=1}(r) Z^{c=1}_{\mathrm{orb}}(r')\,, \tag{4.27}$$

where $Z^{c=1}$ and $Z^{c=1}_{\mathrm{orb}}$ are given in (2.14) and (4.16), respectively. If we mod out by R_2 instead of R_1, by (4.21) we use $\tau_2 \mapsto 1/\tau_2$, i.e. the radii r and r' are interchanged in (4.27). Application of T-duality to both τ and ρ simultaneously, which will be denoted by

$$\mathcal{S}:\ (\tau,\rho)\mapsto\left(-\frac{1}{\tau},-\frac{1}{\rho}\right)$$

and called simultaneous T-duality in the following, amounts to $r\mapsto 1/r$, $r'\mapsto 1/r'$ in both cases, leaving (4.27) invariant. Mirror symmetry $\tau_2\leftrightarrow\rho_2$ acts by $r\mapsto 1/r$, $r'\mapsto r'$, which (4.27) is invariant under, too.

By (4.8) the general reflection orbifold partition function can be written as

$$Z_{R_1-\text{orb}}=\frac{1}{2}\left({}_{1}\underset{1}{\square}+{}_{R_1}\underset{1}{\square}+{}_{1}\underset{R_1}{\square}+{}_{R_1}\underset{R_1}{\square}\right). \tag{4.28}$$

As explained in our general discussion for lattices 6 to 17, the second term in (4.28) gets contributions only from the states built by an action of creation operators on such Hilbert space ground states that are invariant under the action of R_1. The corresponding charge vectors are given in (4.23), namely for lattice 6 with $\tau_1=0,\rho_1=1/2$ we obtain

$$\overset{(-)}{p}=\frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}n\tau_2\pm 2m\rho_2\\ 0\end{pmatrix}=\begin{pmatrix}\frac{n}{r}\pm mr\\ 0\end{pmatrix},\qquad m,n\in\mathbb{Z},r:=\sqrt{\frac{2\rho_2}{\tau_2}}. \tag{4.29}$$

Therefore for the untwisted sector partition function

$$Z_u\left(0,\tau_2,\frac{1}{2},\rho_2\right)=\frac{1}{2}\left(Z+\left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right|\sum_{m,n}q^{\frac{1}{2}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{n}{r}-mr\right)^2}\right).$$

The twisted sector partition function can be calculated by modular transformations, and the complete reflection orbifold partition function is

$$\begin{aligned}Z_{R_1-\text{orb}}(0,\tau_2,\frac{1}{2},\rho_2)=\frac{1}{2}\Bigg(&Z+\left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right|\sum_{m,n}q^{\frac{1}{2}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{n}{r}-mr\right)^2}+\\ &+\left|\frac{\vartheta_3\vartheta_2}{2\eta^4}\right|\sum_{m,n}q^{\frac{1}{8}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{8}\left(\frac{n}{r}-mr\right)^2}+\\ &+\left|\frac{\vartheta_4\vartheta_2}{2\eta^4}\right|\sum_{m,n}(-1)^{mn}q^{\frac{1}{8}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{8}\left(\frac{n}{r}-mr\right)^2}\Bigg),\end{aligned} \tag{4.30}$$

with $r=\sqrt{2\rho_2/\tau_2}$, and for R_2 instead of R_1 with $r=\sqrt{2\tau_2\rho_2}$ by (4.21). Simultaneous T-duality $\mathcal{S}$ amounts to $r\mapsto 1/r$ in both cases. This obviously leaves (4.30) invariant.

Mirror symmetry $U:(\tau,\rho)\mapsto(\rho,\tau)$ commutes with the R_1,R_2 actions on a toroidal theory, so

$$Z_{R-\text{orb}}\left(\frac{1}{2},\tau_2,0,\rho_2\right)=Z_{R-\text{orb}}\left(0,\rho_2,\frac{1}{2},\tau_2\right). \tag{4.31}$$

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Hence the partition function $Z_{R-\mathrm{orb}}(1/2,\tau_2,0,\rho_2)$ for lattice 7 with $\rho_1=0$ is given by (4.30), but now with $r=\sqrt{\rho_2/2\tau_2}$ for R_1, $r=\sqrt{2\tau_2\rho_2}$ for R_2. Again, $\mathcal{S}$ acts by $r\mapsto 1/r$ in both cases and leaves the partition function invariant.

The case $\tau_1=\rho_1=1/2$ (lattice 7) is more subtle. The charge lattice (2.12) of the toroidal theory is generated by the four vectors

$$v_{\delta,\epsilon}:=\frac{1}{2\sqrt{2\tau_2\rho_2}}\left(\begin{pmatrix}\tau_2+\delta\rho_2\\ \epsilon\,(1/2-2\delta\tau_2\rho_2)\end{pmatrix},\begin{pmatrix}\tau_2-\delta\rho_2\\ \epsilon\,(1/2+2\delta\tau_2\rho_2)\end{pmatrix}\right),\qquad \delta,\epsilon\in\{\pm1\}\,,$$

which are pairwise interchanged by R_1 ($v_{\delta,1}\leftrightarrow v_{\delta,-1}$). Denote the corresponding vertex operators by $V(\pm v_{\delta,\epsilon})$. The R_1 invariant part of the charge lattice by (4.23) is given by

$$\overset{(-)}{p}=\frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}n_2\tau_2\pm m_2\rho_2\\ 0\end{pmatrix}=\sqrt{2}\begin{pmatrix}\frac{n}{r}\pm mr\\ 0\end{pmatrix}, \tag{4.32}$$

$$n_2=2n,\quad m_2=2m,\quad n_1=n+m\in\mathbb{Z},\quad r=\sqrt{\frac{\rho_2}{\tau_2}}\,.$$

Because $\langle v_{\delta,\epsilon},v_{\delta,-\epsilon}\rangle=1$, the vertex operators corresponding to generators of the invariant part of the charge lattice are obtained from operator product expansions

$$(V(v_{\delta,1})+V(-v_{\delta,1}))\times(V(v_{\delta,-1})-V(-v_{\delta,-1}))\,.$$

Since this is a product between an R_1 even and an R_1 odd operator, the resulting vertex operators are R_1 odd. It follows that R_1 acts on ground states corresponding to invariant charge vectors (4.32) by $|m_1,m_2,n_1,n_2\rangle\mapsto(-1)^{n_2}|m_1,m_2,n_1,n_2\rangle$. Thus for the untwisted sector partition function we find

$$Z_u\left(\frac12,\tau_2,\frac12,\rho_2\right)=\frac12\left(Z+\left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right|\left(\sum_{m,n\in\mathbb{Z}}-\sum_{m,n\in\mathbb{Z}+1/2}\right)q^{\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\left(\frac{n}{r}-mr\right)^2}\right),$$

with $r=\sqrt{\rho_2/\tau_2}$. We remark that although not stated explicitly above, one may check that in none of the other cases of R_1 actions such additional signs on Hilbert space ground states occur. Here, they are due to the fact that the action of R_1 effectively shifts the B-field by an integer form, as was already mentioned above. In the discussion of the bicritical point (C14) we will point out a very natural confirmation of the above result. By applying modular transformations to ${}_{R_1}\underset{1}{\square}$, we find

$$\begin{aligned}Z_{R_1-\mathrm{orb}}\left(\frac12,\tau_2,\frac12,\rho_2\right)=\frac12\Bigg(&Z+\left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right|\left(\sum_{m,n\in\mathbb{Z}}-\sum_{m,n\in\mathbb{Z}+1/2}\right)q^{\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\left(\frac{n}{r}-mr\right)^2}+\\ &+\left|\frac{\vartheta_3\vartheta_2}{4\eta^4}\right|\sum_{m,n\in\mathbb{Z}}\left(1-(-1)^{n+m}\right)q^{\frac{1}{16}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{16}\left(\frac{n}{r}-mr\right)^2}+\\ &+\left|\frac{\vartheta_4\vartheta_2}{4\eta^4}\right|\sum_{m,n\in\mathbb{Z}}\left(1-(-1)^{n+m}\right)(i)^{nm}q^{\frac{1}{16}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{16}\left(\frac{n}{r}-mr\right)^2}\Bigg),\end{aligned} \tag{4.33}$$

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where $r = \sqrt{\rho_2/\tau_2}$ for the reflection R_1, and therefore $r = 2\sqrt{\tau_2\rho_2}$ for the reflection R_2 by (4.21). To apply simultaneous T-duality $\mathcal{S}$ amounts to $r \mapsto 1/r$, yielding (4.33) invariant. Invariance under mirror symmetry $\tau_2 \leftrightarrow \rho_2$ is also obvious.

By our general discussion for lattices 6 to 17, to find the correct parameter space for the irreducible components of $\mathcal{C}^2$ obtained by $\mathbb{Z}_2(R)$ orbifolding, the Teichmüller spaces are constructed by considering $R = R_1$ only. T-duality applied to τ alone, which by (4.21) is equivalent to $R_1 \leftrightarrow R_2$, does not generically map onto an isomorphic theory. Our calculations above show that simultaneous T-duality $\mathcal{S}$ actually identifies isomorphic theories (see (4.27), (4.30) and (4.33)) as well as mirror symmetry. In particular, lattice 6 ($\tau_1 = 0$) with $\rho_1 = 1/2$ and lattice 7 ($\tau_1 = 1/2$) with $\rho_1 = 0$ correspond to families of isomorphic orbifold conformal field theories.

Summarizing, we have constructed the following three irreducible components of the moduli space:

$$\mathcal{C}^{(0,0)}_{\mathbb{Z}_2(R)-\mathrm{orb}} \cong \left(\mathbb{R}^+\right)^2/\{U,\mathcal{S}\}\,, \qquad \mathcal{C}^{(\frac{1}{2},\frac{1}{2})}_{\mathbb{Z}_2(R)-\mathrm{orb}} \cong \left(\mathbb{R}^+\right)^2/\{U,\mathcal{S}\}$$
$$\mathcal{C}^{(0,\frac{1}{2})}_{\mathbb{Z}_2(R)-\mathrm{orb}} = \mathcal{C}^{(\frac{1}{2},0)}_{\mathbb{Z}_2(R)-\mathrm{orb}} \cong \left(\mathbb{R}^+\right)^2/\mathcal{S}\,.$$

4.4 Lattices 8 and 9: the D_2 orbifold

Lattices 8 ($\tau_1 = 0$) and 9 ($\tau_1 = 1/2$) have a $D_2 = \{1, A(\pi), R_1, R_2\}$ symmetry. By (4.8) for both $\tau_1, \rho_1 \in \{0, 1/2\}$ the D_2 orbifold partition function is

$$\begin{aligned} Z_{D_2-\mathrm{orb}}(\tau_1,\tau_2,\rho_1,\rho_2) &= \frac{1}{4}\sum_{g,h\in D_2} {}_{g}\underset{h}{\square} \\ &= \frac{1}{4}\Bigg(2Z_{\mathbb{Z}_2-\mathrm{orb}} + 2Z_{R_1-\mathrm{orb}} + 2Z_{R_2-\mathrm{orb}} - 2Z + \\ &\quad + {}_{R_1}\underset{A(\pi)}{\square} + {}_{A(\pi)}\underset{R_1}{\square} + {}_{R_2}\underset{R_1}{\square} + {}_{R_2}\underset{A(\pi)}{\square} + {}_{A(\pi)}\underset{R_2}{\square} + {}_{R_1}\underset{R_2}{\square}\Bigg)\,, \end{aligned} \tag{4.34}$$

where we have subtracted $Z = {}_{1}\underset{1}{\square}$ from the second and third term in the second line to avoid overcounting the contribution of the identity element which appears in each reflection group. Observe that by (4.21) separate T-duality (4.26) on τ, or by mirror symmetry equivalently on ρ, interchanges $\mathbb{Z}_2(R_1)$ and $\mathbb{Z}_2(R_2)$. Therefore it maps isomorphic D_2 orbifold conformal field theories onto each other.

The terms in the third line of (4.34) form a modular orbit. To determine them we compute ${}_{R_1}\underset{A(\pi)}{\square}$. Denote by $\mathcal{H}_{A(\pi)}$ the twisted sector Hilbert space of the ordinary $\mathbb{Z}_2$ orbifold which by (4.5) corresponds to fields φ with half integer modes and $\varphi(z=0) = q_j$, $j \in \{1,2,3,4\}$, a $\mathbb{Z}_2$ fixed point on $\mathbb{T}^2$. Assume that k of the four corresponding $\mathbb{Z}_2$ twisted ground states are eigenstates of R_1. There eigenvalues must agree and be ± 1 in order for the $\mathbb{Z}_2$ action on the twisted sector to be well defined. Since by (4.6) the twisted ground states have dimensions $(h,\bar h)=(1/8,1/8)$, we find

$$
\begin{aligned}
{}_{R_1}\underset{A(\pi)}{\square} &= \mathrm{tr}_{\mathcal{H}_{A(\pi)}}\, R_1 q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{c}{24}} \\
&= \pm k\cdot (q\bar q)^{-1/12}\frac{(q\bar q)^{1/8}}{\prod_{n=1}^{\infty}(1-q^{n-1/2})(1-\bar q^{n-1/2})(1+q^{n-1/2})(1+\bar q^{n-1/2})} \\
&= \pm k\left|\frac{\eta^2}{\vartheta_3\vartheta_4}\right| = \pm\frac{k}{2}\left|\frac{\vartheta_2}{\eta}\right|. \qquad (4.35)
\end{aligned}
$$

All in all by modular transformations the third line in (4.34) is equal to $\pm 2kZ_{\text{Ising}}$, where

$$
Z_{\text{Ising}}=\frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|+\left|\frac{\vartheta_4}{\eta}\right|+\left|\frac{\vartheta_3}{\eta}\right|\right) \qquad (4.36)
$$

and $k\in\{0,2,4\}$, because R_1 must map twisted ground states onto twisted ground states. To determine the correct factor k we first note that in case $\tau_1=\rho_1=0$ the original toroidal theory decomposes into a tensor product of two $c=1$ theories. The action of D_2 respects the product structure, hence

$$
Z_{D_2^+-\text{orb}}(0,\tau_2,0,\rho_2)=Z_{\text{orb}}^{c=1}(\sqrt{\tau_2\rho_2})Z_{\text{orb}}^{c=1}(\sqrt{\rho_2/\tau_2})\,, \qquad (4.37)
$$

where $Z_{\text{orb}}^{c=1}$ was given in (4.16). One now checks that in this case $k=4$, in agreement with the geometric observation that all the four $\mathbb{Z}_2$ fixed points on $\mathbb{T}^2$ are invariant under the R actions. For $\tau_1=1/2,\rho_1=0$ one can argue that only two of the four fixed points are invariant, thus $k=2$. If $\rho_1=1/2$, this geometric argument breaks down since, as noted in our general discussion for lattices 6 to 17, in this case the symmetries R_1,R_2 effectively shift the B-field by an integer form. The correct factor for $\tau_1=0,\rho_1=1/2$ is $k=2$, as well. This follows from the construction of the D_4 orbifold conformal field theory (lattice 15), where we will see that the D_2 orbifold at $\tau_1=0,\rho_1=1/2$ must always contain an even number of fields with dimensions $h=\bar h=1/16$. For $\tau_1=\rho_1=1/2$ we find $k=0$. This follows from the fact that ${}_{1}\underset{R_1}{\square}$ by (4.33) generically does not get any contributions from fields with dimensions $h=\bar h=1/16$. Hence ${}_{A(\pi)}\underset{R_1}{\square}=\pm\frac{k}{2}\left|\frac{\vartheta_3}{\eta}\right|$ cannot give such contributions either. In summary,

$$
\begin{aligned}
Z_{D_2^\pm-\text{orb}}(\tau_1,\tau_2,\rho_1,\rho_2) &= \frac{1}{2}(Z_{\mathbb{Z}_2-\text{orb}}+Z_{R_1-\text{orb}}+Z_{R_2-\text{orb}}\pm kZ_{\text{Ising}}-Z)\,, \\
k &= 4(1-\tau_1-\rho_1)\,, \qquad (4.38)
\end{aligned}
$$

where $Z_{\mathbb{Z}_2-\text{orb}}$ is given in (4.15), and $Z_{R-\text{orb}}$ is given in (4.27), (4.30) or (4.33), respectively. In particular, for this orbifold construction discrete torsion has a nontrivial effect, and we can produce two non-equivalent theories corresponding to lattice 8 and each possible value of ρ_1. We stress that we have been discussing a perhaps counterintuitive effect of "turning on the B-field": the action of R_1,R_2 on twisted ground states depends severely on the value of ρ_1. In particular, they must not be interpreted from a purely geometric point of view.

Because $\mathbb{Z}_2(R)$ orbifold conformal field theories as well as the formula for k in (4.38) are invariant under mirror symmetry, the same is true for D_2 orbifolds. Hence we have constructed five irreducible components of $\mathcal{C}^2$,

$$\mathcal{C}^{(0,0)}_{D_2^{\pm}-\mathrm{orb}} \cong \left(\mathbb{R}^+/\Theta\right)^2/U, \quad \mathcal{C}^{(0,\frac{1}{2})}_{D_2^{\pm}-\mathrm{orb}} \cong \mathcal{C}^{(\frac{1}{2},0)}_{D_2^{\pm}-\mathrm{orb}} \cong \left(\mathbb{R}^+/\Theta\right)^2, \quad \mathcal{C}^{(\frac{1}{2},\frac{1}{2})}_{D_2-\mathrm{orb}} \cong \left(\mathbb{R}^+/\Theta\right)^2/U.$$

4.5 Lattice 10: the $\mathbb{Z}_2(T_R)$ reflection plus shift orbifold

Lattice 10 ($\tau_1 = 0$) has reflection plus shift symmetry $\mathbb{Z}_2(T_{R_1}) = \{1, T_{R_1} = R_1 e^{2\pi i p\cdot\delta_1/\sqrt{2}}\}$, where $\delta_1 = \sqrt{\rho_2/\tau_2}(1/2, 0)$. From our general discussion on the $T_R^{(\prime)}$ action for lattices 6 to 17 we know that we only have to consider the case $\rho_1 = 0$. By (4.8) the general reflection plus shift orbifold partition function is

$$Z_{T_{R_1}-\mathrm{orb}} = \frac{1}{2}\left({}_{1}\underset{1}{\square} + {}_{T_{R_1}}\underset{1}{\square} + {}_{1}\underset{T_{R_1}}{\square} + {}_{T_{R_1}}\underset{T_{R_1}}{\square}\right). \tag{4.39}$$

The torus theory is a tensor product of two $c = 1$ circle theories (2.15): ${}_{1}\underset{1}{\square} = {}_{1}\underset{1}{\square}(\varphi_1)\,{}_{1}\underset{1}{\square}(\varphi_2)$. The $\mathbb{Z}_2(T_{R_1})$ action respects the product structure, therefore we have ${}_{T_{R_1}}\underset{1}{\square} = {}_{t_\epsilon}\underset{1}{\square}(\varphi_1)\,{}_{(-1)}\underset{1}{\square}(\varphi_2)$ with $t_\epsilon = e^{2\pi i p\cdot\frac{\epsilon}{\sqrt{2}}}$, $\epsilon = \frac{1}{2}\sqrt{\rho_2/\tau_2}$. From the circle orbifold theory (4.16) one has ${}_{(-1)}\underset{1}{\square}(\varphi_2) = 2\,|\eta/\vartheta_2|$. As explained in our general discussion for lattices 6 to 17, the translation symmetry t_ϵ does not affect oscillator modes. By (4.23) it acts on the Hilbert space ground states $|0, m, 0, n\rangle$ of the circle theory ${}_{1}\underset{1}{\square}(\varphi_1)$ via multiplication with $(-1)^n$. So using (2.14) and (4.23) with $r := \sqrt{\rho_2/\tau_2}$ we find

$${}_{t_\epsilon}\underset{1}{\square}(\varphi_1) = \frac{1}{\eta\bar{\eta}}\sum_{m,n}(-1)^n q^{\frac{1}{4}(\frac{n}{r}+mr)^2}\bar{q}^{\frac{1}{4}(\frac{n}{r}-mr)^2}.$$

The remaining boxes in (4.39) are obtained by modular transformations. Thus the complete partition function is

$$\begin{aligned} Z_{T_{R_1}-\mathrm{orb}}(0,\tau_2,0,\rho_2) = \frac{1}{2}\Bigg(Z &+ \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n}(-1)^n q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2} + \\ &+ \left|\frac{\vartheta_3\vartheta_2}{\eta^4}\right| \sum_{n\in\mathbb{Z}, m\in\mathbb{Z}+1/2} q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2} + \\ &+ \left|\frac{\vartheta_4\vartheta_2}{\eta^4}\right| \sum_{n\in\mathbb{Z}, m\in\mathbb{Z}+1/2} (-1)^n q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\bar{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2}\Bigg), \end{aligned} \tag{4.40}$$

where $r := \sqrt{\rho_2/\tau_2}$. If we mod out by $\mathbb{Z}_2(T_{R_2})$ instead of $\mathbb{Z}_2(T_{R_1})$ by (4.24) we have to set $r := \sqrt{\tau_2\rho_2}$. Simultaneous T-duality $\mathcal{S}$ amounts to $r \mapsto 1/r$ which does not leave (4.40) invariant. But for $\mathbb{Z}_2(T_{R_1})$ orbifolds, $\mathcal{S}$ combined with mirror symmetry maps onto an isomorphic theory, whereas for $\mathbb{Z}_2(T_{R_2})$ orbifolds mirror symmetry maps onto isomorphic theories. Therefore the $\mathbb{Z}_2(T_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{\mathbb{Z}_2(T_R)-\mathrm{orb}} \cong \left(\mathbb{R}^+\right)^2 / U \,.$$

4.6 Lattice 11: the $D_2(T_R)$ orbifold

Lattice 11 ($\tau_1 = 0$) has a $D_2(T_R) = \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}$ symmetry as defined in (3.2), and we can generally set $\rho_1 = 0$. By (4.8) the partition function has the form

$$\begin{aligned} Z_{D_2(T_R)-\mathrm{orb}}(0,\tau_2,0,\rho_2) = \frac{1}{4}\Bigg(& 2Z_{\mathbb{Z}_2-\mathrm{orb}} + 2Z_{T_{R_1}-\mathrm{orb}} + 2Z_{\widehat{T}_{R_2}-\mathrm{orb}} - 2Z + \\ & + {}_{T_{R_1}}\underset{A(\pi)}{\square} + {}_{\widehat{T}_{R_2}}\underset{A(\pi)}{\square} + {}_{A(\pi)}\underset{T_{R_1}}{\square} + {}_{A(\pi)}\underset{\widehat{T}_{R_2}}{\square} + {}_{T_{R_1}}\underset{\widehat{T}_{R_2}}{\square} + {}_{\widehat{T}_{R_2}}\underset{T_{R_1}}{\square} \Bigg). \end{aligned} \tag{4.41}$$

The terms in the second line can be computed by a similar argument as those in the third line of (4.34). Only here none of the four ordinary $\mathbb{Z}_2$ fixed points is invariant under T_{R_1} or $\widehat{T}_{R_2}$, so the first two boxes vanish. The others are obtained by modular transformations from these and therefore vanish as well. In particular, in this case discrete torsion has no effect on the partition function.

The original toroidal theory decomposes into the tensor product of two $c = 1$ theories (2.15). By (3.2) $\widehat{T}_{R_2}$ leaves the second factor invariant. Since on the Hilbert space ground states $|0, m, 0, n\rangle$ of the first factor $\widehat{T}_{R_2}|0, m, 0, n\rangle = \pm|0, -m, 0, -n\rangle = \pm R_2|0, m, 0, n\rangle$, ${}_{\widehat{T}_{R_2}}\underset{1}{\square} = {}_{R_2}\underset{1}{\square}$ and therefore we have $Z_{\widehat{T}_{R_2}-\mathrm{orb}} = Z_{R_2-\mathrm{orb}}$. All in all

$$Z_{D_2(T_R)-\mathrm{orb}}(0,\tau_2,0,\rho_2) = \frac{1}{2}\left(Z_{\mathbb{Z}_2-\mathrm{orb}} + Z_{R_2-\mathrm{orb}} + Z_{T_{R_1}-\mathrm{orb}} - Z\right), \tag{4.42}$$

where $Z_{\mathbb{Z}_2-\mathrm{orb}}, Z_{R_2-\mathrm{orb}}, Z_{T_{R_1}-\mathrm{orb}}$ are given in (4.15), (4.27), and (4.40), respectively.

By the discussion of $\mathbb{Z}_2(T_R)$ orbifold conformal field theories (lattice 10), only combined $\mathcal{S}$ with mirror symmetry leaves $Z_{T_{R_1}-\mathrm{orb}}$ invariant. This also maps isomorphic R orbifolds onto each other (lattice 6), so the $D_2(T_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{D_2(T_R)-\mathrm{orb}} \cong \left(\mathbb{R}^+\right)^2 / U\mathcal{S}.$$

4.7 Lattice 12: the $D_2(\mathbb{T}'_R)$ orbifold

Lattice 12 ($\tau_1 = 0$) has a $D_2(\mathbb{T}'_R) = \{1, A(\pi), \mathbb{T}'_{R_1}, \mathbb{T}'_{R_2}\}$ symmetry as defined in (3.2), and we may set $\rho_1 = 0$. The calculation of the partition functions is analogous to that for lattice 11, where in (4.41) we now replace T_{R_1} by $\mathbb{T}'_{R_1}$ and $\widehat{T}_{R_2}$ by $\mathbb{T}'_{R_2}$. Again, none of the ordinary $\mathbb{Z}_2$ fixed points is invariant under a symmetry T'_R. So the second line in $(4.41)'$ vanishes, too, and discrete torsion has no effect.

For $\tau_1 = \rho_1 = 0$ analogously to $Z_{\widehat{T}_{R_2}-\text{orb}} = Z_{R_2-\text{orb}}$ in the partition function for lattice 11 we now find $Z_{\mathbb{T}'_R-\text{orb}} = Z_{T_R-\text{orb}}$. So we have

$$Z_{D_2(\mathbb{T}'_R)-\text{orb}}(0,\tau_2,0,\rho_2) = \frac{1}{2}(Z_{\mathbb{Z}_2-\text{orb}} + Z_{T_{R_1}-\text{orb}} + Z_{T_{R_2}-\text{orb}} - Z)\,, \tag{4.43}$$

where $Z_{\mathbb{Z}_2-\text{orb}}$ and $Z_{T_R-\text{orb}}$ are given in (4.15) and (4.40), respectively. Since T-duality (4.24) applied to τ interchanges $\mathbb{Z}_2(T_{R_1})$ and $\mathbb{Z}_2(T_{R_2})$, but neither simultaneous T-duality $\mathcal{S}$ nor mirror symmetry leaves invariant both of them, $D_2(T'_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{D_2(\mathbb{T}'_R)-\text{orb}} \cong \left(\mathbb{R}^+/\Theta\right) \times \mathbb{R}^+\,.$$

4.8 Lattice 13: the $D_3(R_1)$ orbifold

Lattice 13 ($\tau = e^{2\pi i/3}$) has a $D_3(R_1) = \mathbb{Z}_3 \cup \{R_1, A(2\pi/3)R_1, A(4\pi/3)R_1\}$ symmetry. By our general discussion for lattices 6 to 17 and since for lattice 13 the value of τ_2 must be fixed to $\tau_2 = \sqrt{3}/2$, the components of the moduli space $\mathcal{C}^2$ obtained by $D_3(R_1)$ orbifolding are

$$\mathcal{C}^{(\rho_1)}_{D_3(R_1)-\text{orb}} \cong \mathbb{R}^+\,, \qquad \rho_1 \in \left\{0, \frac{1}{2}\right\}.$$

The maximal abelian subgroups of $D_3(R_1)$ are $\mathbb{Z}_3$, and three order two groups $\{1, R_1\}$,$\{1, A(2\pi/3)R_1\}$, $\{1, A(4\pi/3)R_1\}$. These groups give identical contributions to the partition function since they are conjugate within $D_3(R_1)$. Using (4.10) we therefore find

$$\begin{aligned} Z_{D_3(R_1)-\text{orb}}\left(1/2, \sqrt{3}/2, \rho_1, \rho_2\right) &= \frac{1}{6}\left(3Z_{\mathbb{Z}_3-\text{orb}} + 3(2Z_{R_1-\text{orb}} - Z)\right) \\ &= \frac{1}{2}\left(Z_{\mathbb{Z}_3-\text{orb}} + 2Z_{R_1-\text{orb}} - Z\right), \end{aligned} \tag{4.44}$$

where $Z_{\mathbb{Z}_3-\text{orb}}$ is given in (4.18), and $Z_{R_1-\text{orb}}$ is given in (4.31) or in (4.33) for $\rho_1 = 0$ or $\rho_1 = 1/2$, respectively.

4.9 Lattice 14: The $D_3(R_2)$ orbifold

Lattice 14 ($\tau = e^{2\pi i/3}$) has a $D_3(R_2) = \mathbb{Z}_3 \cup \{R_2, A(2\pi/3)R_2, A(4\pi/3)R_2\}$ symmetry. Analogously to lattice 13 we find

$$Z_{D_3(R_2)-\text{orb}}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \rho_1, \rho_2\right) = \frac{1}{2}\left(Z_{\mathbb{Z}_3-\text{orb}} + 2Z_{R_2-\text{orb}} - Z\right), \tag{4.45}$$

where $Z_{\mathbb{Z}_3-\text{orb}}$ is given in (4.18), and $Z_{R_2-\text{orb}}$ is given in (4.31) and (4.33) for $\rho_1 = 0$ or $\rho_1 = 1/2$, respectively.

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From our discussion of lattices 6 and 7 we know that $Z_{R_2-\mathrm{orb}}$ is obtained from $Z_{R_1-\mathrm{orb}}$ by application of T-duality (4.21) on τ. Using mirror symmetry we see that we can equally apply T-duality to ρ and find

$$
\begin{aligned}
Z_{D_3(R_2)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},0,\rho_2\right) &= Z_{D_3(R_1)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},0,\frac{1}{\rho_2}\right),\\
Z_{D_3(R_2)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\rho_2\right) &= Z_{D_3(R_1)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{1}{4\rho_2}\right).
\end{aligned}
$$

The above actually is the equation for T-duality on $\mathcal{C}^{(\rho_1)}_{D_3(R_1)-\mathrm{orb}}$. In particular, the $D_3(R_2)$ orbifold procedure does not yield a new component of the moduli space $\mathcal{C}^2$ but only reproduces $\mathcal{C}^{(\rho_1)}_{D_3(R_1)-\mathrm{orb}}$, $\rho_1 \in \{0, 1/2\}$.

4.10 Lattice 15: the D_4 orbifold

Lattice 15 ($\tau = i$) has a $D_4 = \mathbb{Z}_4 \cup \{R_1, A(\pi/2)R_1, R_2, A(\pi/2)R_2\}$ symmetry. The maximal abelian subgroups of D_4 are $\mathbb{Z}_4$, $D_2 = \{1, A(\pi), R_1, R_2\}$, and $D_2' = \{1, A(\pi), A(\pi/2)R_1, A(\pi/2)R_2\}$. The two order four groups D_2 and D_2' give different contributions to the partition function, since these groups are not conjugate in D_4. The fundamental cells of lattice 15 we have to pick in order to interprete them as reflections along the edges of the cell have different shape. For D_2 it is a unit square giving a contribution $Z_{D_2-\mathrm{orb}}(\tau = i, \rho)$, whereas for D_2' it is a rhombus giving a contribution $Z_{D_2-\mathrm{orb}}(\tau = 1/2 + i/2, \rho)$. Note that by (4.38) for $\rho_1 = 0$ we have an independent choice of sign for the discrete torsion parts of D_2, D_2', and for $\rho_1 = 1/2$ discrete torsion enters for D_2 only. Using (4.10) and $\delta, \epsilon \in \{\pm\}$ for the partition function we therefore get

$$
\begin{aligned}
Z_{D_4^{\delta\epsilon}-\mathrm{orb}}(0,1,0,\rho_2) &= \frac{1}{2}\Bigg(Z_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,\rho_2) + Z_{D_2^{\delta}-\mathrm{orb}}(0,1,0,\rho_2) + \\
&\qquad + Z_{D_2^{\epsilon}-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,\rho_2\right) - Z_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,\rho_2)\Bigg),\\
Z_{D_4^{\pm}-\mathrm{orb}}\left(0,1,\frac{1}{2},\rho_2\right) &= \frac{1}{2}\Bigg(Z_{\mathbb{Z}_4-\mathrm{orb}}\left(0,1,\frac{1}{2},\rho_2\right) + Z_{D_2^{\pm}-\mathrm{orb}}\left(0,1,\frac{1}{2},\rho_2\right) + \\
&\qquad + Z_{D_2-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\rho_2\right) - Z_{\mathbb{Z}_2-\mathrm{orb}}\left(0,1,\frac{1}{2},\rho_2\right)\Bigg),
\end{aligned}
\tag{4.46}
$$

where $Z_{\mathbb{Z}_2-\mathrm{orb}}$, $Z_{\mathbb{Z}_4-\mathrm{orb}}$ and $Z_{D_2^{\pm}-\mathrm{orb}}$ are given in (4.15), (4.19), and (4.38). We remark that in case $\rho_1 = 1/2$ the $Z_{\mathbb{Z}_4}, Z_{D_2}, Z_{\mathbb{Z}_2}$ parts of (4.46) always contribute even numbers of fields with dimensions $h = \bar{h} = 1/16$. This shows that for the D_2 orbifold with $\tau_1 = 0, \rho_1 = 1/2$ we must indeed have $k = 2$ in (4.38).

Since for the D_2 orbifold by our discussion of lattices 8 and 9 separate T-duality may be performed on τ, ρ without changing the theory, the D_4 orbifold conformal

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field theories form six families

$$\mathcal{C}^{(0)}_{D_4^{\delta\epsilon}-\mathrm{orb}} \cong \mathbb{R}^+/\Theta\,, \quad \delta,\epsilon \in \{\pm\}\,, \qquad \mathcal{C}^{(1/2)}_{D_4^{\pm}-\mathrm{orb}} \cong \mathbb{R}^+/\Theta\,.$$

4.11 Lattice 16: the $D_4(\mathbb{T}'_R)$ orbifold

Lattice 16 ($\tau = i$) has a $D_4(\mathbb{T}'_R) = \mathbb{Z}_4 \cup \{\mathbb{T}'_{R_1}, A(\pi/2)\mathbb{T}'_{R_1}, \mathbb{T}'_{R_2}, A(\pi/2)\mathbb{T}'_{R_2}\}$ symmetry as defined in (3.2), and we may set $\rho_1 = 0$. The maximal abelian subgroups of $D_4(T'_R)$ are $\mathbb{Z}_4$, $D_2(\mathbb{T}'_R) = \{1, A(\pi), \mathbb{T}'_{R_1}, \mathbb{T}'_{R_2}\}$, and $D_2 = \{1, A(\pi), A(\pi/2)\mathbb{T}'_{R_1}, A(\pi/2)\mathbb{T}'_{R_2}\}$. Anologously to lattice 15 we find

$$\begin{aligned} Z_{D_4(\mathbb{T}'_R)^{\pm}-\mathrm{orb}}(0,1,0,\rho_2) = \frac{1}{2}\Bigg(& Z_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,\rho_2) + Z_{D_2(\mathbb{T}'_R)-\mathrm{orb}}(0,1,0,\rho_2) + \\ & + Z_{D_2^{\pm}-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,\rho_2\right) - Z_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,\rho_2)\Bigg), \end{aligned} \tag{4.47}$$

where $Z_{\mathbb{Z}_2-\mathrm{orb}}$, $Z_{\mathbb{Z}_4-\mathrm{orb}}$, $Z_{D_2(\mathbb{T}'_R)-\mathrm{orb}}$ and $Z_{D_2^{\pm}-\mathrm{orb}}$ are given in (4.15), (4.19), (4.43), and (4.38). The discussion of the $D_2(T'_R)$ orbifold (lattice 12) shows that T-duality does not map equivalent $D_4(T'_R)$ orbifold theories onto each other, thus

$$\mathcal{C}_{D_4(T'_R)^{\pm}-\mathrm{orb}} \cong \mathbb{R}^+\,.$$

4.12 Lattice 17: the D_6 orbifold

Lattice 17 ($\tau = e^{2\pi i/3}$) has a $D_6 = \mathbb{Z}_6 \cup \{R_1, A(\pi/3)R_1, A(2\pi/3)R_1, R_2, A(\pi/3)R_2, A(2\pi/3)R_2\}$ symmetry. The maximal abelian subgroups of D_6 are $\mathbb{Z}_6$, and three groups of type D_2, namely $\{1, A(\pi), R_1, R_2\}$, $\{1, A(\pi), A(\pi/3)R_1, A(\pi/3)R_2\}$, $\{1, A(\pi), A(2\pi/3)R_1, A(2\pi/3)R_2\}$. These order four groups give identical contributions to the partition function since they are conjugate in D_6. This also means that in order for the action of D_6 on the twisted sector to be well defined, discrete torsion must be the same for all the three of them. Now by (4.10) the complete partition function is

$$\begin{aligned} Z_{D_6^{(\pm)}-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\rho_1,\rho_2\right) &= \frac{1}{12}\left(6Z_{\mathbb{Z}_6-\mathrm{orb}} + 3\left(4Z_{D_2^{(\pm)}-\mathrm{orb}} - 2Z_{\mathbb{Z}_2-\mathrm{orb}}\right)\right) \\ &= \frac{1}{2}\left(Z_{\mathbb{Z}_6-\mathrm{orb}} + 2Z_{D_2^{(\pm)}-\mathrm{orb}} - Z_{\mathbb{Z}_2-\mathrm{orb}}\right), \end{aligned} \tag{4.48}$$

where $Z_{\mathbb{Z}_2-\mathrm{orb}}$, $Z_{\mathbb{Z}_6-\mathrm{orb}}$ and $Z_{D_2^{(\pm)}-\mathrm{orb}}$ are given in (4.15), (4.20), and (4.38). Analogously to lattice 15, the three families of D_6 orbifold conformal field theories are

$$\mathcal{C}^{(0)}_{D_6^{\pm}-\mathrm{orb}} \cong \mathbb{R}^+/\Theta\,, \qquad \mathcal{C}^{(1/2)}_{D_6-\mathrm{orb}} \cong \mathbb{R}^+/\Theta\,.$$

5. Multicritical lines and points

We now determine all intersections of the 28 nonexceptional components $\mathcal{C}^{(\bullet)}_{G^{(\bullet)}-\mathrm{orb}}$ of the moduli space that we constructed in section 4. We find that all but four of them can be connected directly or indirectly to the moduli space $\mathcal{T}^2$ of toroidal theories, and $\mathcal{C}^2$ exhibits a complicated structure with various loops.

The procedure closely follows the proof for the isomorphy of the $c = 1$ circle theory at radius $r = 2$ to the orbifold theory at radius $r = 1$ (see, e.g., [8, 13]). The main idea is to exploit the enhanced SU(2) symmetry of the circle theory at radius $r = 1$. Namely, SU(2) relates two generically different $\mathbb{Z}_2$ actions in this theory by conjugation. Thus the resulting orbifold theories are isomorphic. One of them is the circle theory at doubled radius $r = 2$, the other is the ordinary $\mathbb{Z}_2$ orbifold theory at radius $r = 1$.

Using results of B. Rostand's we can show that the generalization of the above procedure to $c = 2$ will suffice to find all intersections of our 29 nonexceptional nonisolated components of $\mathcal{C}^2$. Namely, in [20, 21] it is shown that every multicritical point on the moduli space $\mathcal{T}^2$ of toroidal theories is an orbifold of another toroidal theory with enhanced symmetry. By our discussion in section 2, we may restrict ourselves to the study of left-right symmetric orbifolds. In particular, to find all intersections of $\mathcal{T}^2$ with one of the 28 nonexceptional orbifold components it suffices to determine all toroidal theories with enhanced left and right symmetry (which in the following are simply called theories with enhanced symmetry) and mod out all symmetries which are conjugate to some shift on the charge lattice. As anticipated in [4] each of the toroidal multicritical points generates a series of further multicritical points or lines, since we can mod out further symmetries. But even better, this procedure will lead to the determination of all intersection points: by the discussion in sections 2 and 3, all the 28 nonexceptional components of $\mathcal{C}^2$ are obtained by modding out solvable groups from toroidal theories. This means that we can always regain the original toroidal theory by performing another orbifold procedure. In particular, any intersection point between nonexceptional nonisolated components of $\mathcal{C}^2$ corresponds to a multicritical point on $\mathcal{T}^2$.

One can simplify things by stepwise modding out [6]: if a symmetry group G contains a normal subgroup H, then the G orbifold conformal field theory A/G of a theory A is isomorphic to the G/H orbifold conformal field theory of A/H. Moreover, the G/H action on A/H translates to an action on any other theory A' which was identified with A/H. For $H' \subset G/H, G' \cong H \times H'$ this leads to possibly new identifications $A/G' \cong A'/H'$ which need not correspond to conjugate actions on the original A. In $A/H \cong A'$ we may have gotten rid of all states which the G' action has no consistent conjugate on.

In section 5.1 we start by determining all points of enhanced symmetry in $\mathcal{T}^2$. The idea of proof again is closely related to the techniques used in [20, 21]. In section 5.2 we discuss all the multicritical points and lines obtainable by modding out conjugate $\mathbb{Z}_2$ symmetries of tori with enhanced SU(2) symmetry. In sections 5.3–5.7 we determine all multicritical points and lines obtainable from those identifications we found in 5.2 by modding out further symmetries. Afterwards (section 5.8) we follow the same procedure for the SU(3) torus theory at $\tau = \rho = e^{2\pi i/3}$. The slightly technical discussion results in a list of all multicritical points and lines in nonexceptional nonisolated components of $\mathcal{C}^2$.

We remark that all the identifications below have been confirmed by us on the level of partition functions numerically. We will denote the $G^{(\bullet)}$ orbifold theory of the toroidal theory $A_T(\tau_1, \tau_2, \rho_1, \rho_2)$ with parameters (τ, ρ) by $A_{G^{(\bullet)}-\mathrm{orb}}(\tau_1, \tau_2, \rho_1, \rho_2)$ in the following.

5.1 Points of enhanced symmetry in $\mathcal{T}^2$

Assume that a toroidal conformal field theory with charge lattice Γ has enhanced symmetry. By $\{(\pm p_i, 0), (0, \pm p_i'), i \in \{1, \ldots, d\}\} \subset \Gamma$ we denote the charge vectors corresponding to the additional vertex operators of dimensions $(1,0)$ and $(0,1)$, respectively. In particular, $|p_i|^2 = |p_i'|^2 = 2$, and since the corresponding vertex operators are pairwise local, for $i \neq j$ we may assume $p_i \cdot p_j = p_i' \cdot p_j' \in \{0, 1\}$. Then the $\mathbb{R}$–span of $\{(p_i, p_i'), i \in \{1, \ldots, d\}\} \subset \Gamma$ is totally isotropic with respect to the scalar product (2.8). This means that we may choose a geometric interpretation (Λ, B) of our toroidal theory such that $p_i = p_i'$ for all $i \in \{1, \ldots, d\}$ (see [1, 16]). Moreover, by the above restrictions on the scalar products between the p_i, these vectors generate the root lattice of a simply laced Lie group. Since the rank of this group can be at most two, the only possible groups are $A_2 = \mathrm{SU}(3)$, $A_1^2 = \mathrm{SU}(2)^2$ or $A_1 = \mathrm{SU}(2)$. If we now write the charge vectors $(p_i, 0)$ and $(0, -p_i)$ in the form (2.4), we find

$$\forall\, i \in \{1, \ldots, d\}: \quad \pm p_i = \frac{1}{\sqrt{2}} \left(\mu_i^{\pm} - B\lambda_i \pm \lambda_i \right), \qquad \lambda_i \in \Lambda, \mu_i^{\pm} \in \Lambda^* .$$

In particular, $2\lambda_i, 2B\lambda_i \in \Lambda^*$ for all $i \in \{1, \ldots, d\}$. These conditions suffice to determine all theories in $\mathcal{T}^2$ with (left-right symmetrically) enhanced symmetry.

There are two theories with maximally (i.e. rank two) enhanced symmetry, namely the $\mathrm{SU}(2)^2$ torus theory at $\tau = \rho = i$ and the SU(3) torus theory at $\tau = \rho = e^{2\pi i/3}$. Tori with $\tau = \rho \notin \{i, e^{2\pi i/3}\}$ and $\tau_1 \in \{0, 1/2\}$ exhibit an enhanced SU(2) symmetry.

5.2 Multicritical lines on the torus moduli space $\mathcal{T}^2$: conjugate $\mathbb{Z}_2$ actions

To compare all $\mathbb{Z}_2$ symmetries of the $\mathrm{SU}(2)^2$ torus theory at $\tau = \rho = i$ we discuss their action on the $(1,0)$ fields. As in section 2, the conserved currents of the generic toroidal theory are called j_μ. The additional vertex operators of dimensions $(1,0)$ are denoted $j_\mu^{\pm}, \mu \in \{1, 2\}$, such that each triple $j_\mu, j_\mu^{\pm}$ generates an $\mathrm{SU}(2)_1$ Kac–Moody algebra. Each of these $\mathrm{SU}(2)_1$ Kac–Moody algebras belongs to one of the

$c = 1$ factors of the torus theory. Let us list all $\mathbb{Z}_2$ symmetries with two positive and four negative eigenvalues on the set of $(1, 0)$ fields. By $\widetilde{\mathbb{Z}_2(R)}$ we denote the $\mathbb{Z}_2(R)$ symmetry applied to the torus theory with fundamental cell such that $\tau = \rho = 1/2 + i/2$ (remember the phases on Hilbert space ground states that were discussed for lattice 9):

$$\begin{array}{lll}
\mathbb{Z}_2 \text{ rotational group}: & j_\mu \mapsto -j_\mu, & j_\mu^\pm \leftrightarrow j_\mu^\mp\,, \\
\text{shift orbifold by } \delta' = \frac{1}{2}\binom{1}{1}: & j_\mu \mapsto j_\mu, & j_\mu^\pm \mapsto -j_\mu^\pm\,, \\
\widetilde{\mathbb{Z}_2(R_1)}: & j_1 \mapsto j_1, j_2 \mapsto -j_2, & j_1^\pm \mapsto -j_1^\pm, j_2^+ \leftrightarrow j_2^-\,, \\
\mathbb{Z}_2(T_{R_1}): & j_1 \mapsto j_1, j_2 \mapsto -j_2, & j_1^\pm \mapsto -j_1^\pm, j_2^+ \leftrightarrow j_2^-\,.
\end{array}$$

None of the above symmetries mixes currents from different $c = 1$ factors of the torus theory or j_μ with $j_\mu^\pm$ currents. Moreover, their eigenvalue spectrum is identical on each $c = 1$ factor, so we may use the corresponding $c = 1$ result to show that the four $\mathbb{Z}_2$ orbifolds by the above listed symmetries give isomorphic theories when applied to the $\mathrm{SU}(2)^2$ theory. This generates a quadrucritical point. The shift orbifold by the half lattice vector δ', as usual, results in a torus theory with additional generator δ' of the lattice and half volume and B-field ($A_T(0, 1, 0, 2) = A_T(0, 1, 0, 1/2)$ by T-duality):

$$\begin{aligned} A_T(0,1,0,2) &= A_{T_R-\mathrm{orb}}(0,1,0,1) \\ &= A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1) = A_{R-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). \end{aligned} \tag{Q1}$$

The equality $A_T(0, 1, 0, 2) = A_{\mathbb{Z}_2-\mathrm{orb}}(0, 1, 0, 1)$ has already been proven in [15], both on the level of partition function and operator algebra.

The above quadrucritical point turns out to actually be the intersection of four bicritical lines. First consider the family of torus theories at parameters $\tau = \rho = it, t \in \mathbb{R}^+$ which decompose into tensor products of two $c = 1$ circle theories at radii $r = 1$ and $r' = t$, respectively. For all values of t the first factor possesses an $\mathrm{SU}(2)$ symmetry. Since the actions of T_{R_2} and the shift by $\delta' = \frac{1}{2}\binom{1}{t}$ only differ on this first factor, where they are generally conjugate by the $\mathrm{SU}(2)$ symmetry, we find ($A_T(1/2, t/2, 0, t/2) = A_T(0, t/2, 1/2, t/2)$ by mirror symmetry)

$$\forall\, t \in \mathbb{R}^+ : \quad A_T\left(0, \frac{t}{2}, \frac{1}{2}, \frac{t}{2}\right) = A_{T_{R_2}-\mathrm{orb}}(0,t,0,t)\,, \tag{L1}$$

and analogously

$$\forall\, t \in \mathbb{R}^+ : \quad A_{T_{R_1}-\mathrm{orb}}(0,t,0,t) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,t,0,t)\,. \tag{L2}$$

Next consider the family of toroidal theories at parameters $\tau = \rho = 1/2 + it$, $t \in \mathbb{R}^+$. We also have a generic $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry for this family. Inspection of the charge lattice shows that as before we have conjugate $\mathbb{Z}_2$ symmetries now giving bicritical lines

$$\forall\, t \in \mathbb{R}^+ : \quad A_{\mathbb{Z}_2-\mathrm{orb}}\left(\frac{1}{2}, t, \frac{1}{2}, t\right) = A_{R_1-\mathrm{orb}}\left(\frac{1}{2}, t, \frac{1}{2}, t\right), \tag{L3}$$

$$A_{R_2-\mathrm{orb}}\left(\frac{1}{2}, t, \frac{1}{2}, t\right) = A_T\left(0, 2t, \frac{1}{4}, \frac{t}{2}\right). \tag{L4}$$

There are two more $\mathbb{Z}_2$ symmetries which are conjugate on the entire family of toroidal theories with parameters $\tau = \rho = it, t \in \mathbb{R}^+$ by SU(2) symmetry on the first factor. They have four positive and two negative eigenvalues on $(1,0)$ fields:

$$\begin{array}{llll} \mathbb{Z}_2(R_2): & j_1 \mapsto -j_1\,, \quad j_2 \mapsto j_2\,, & j_1^+ \leftrightarrow j_1^-\,, & j_2^\pm \mapsto j_2^\pm\,, \\ \text{shift orbifold by } \delta_1 = \frac{1}{2}\binom{1}{0}: & j_\mu \mapsto j_\mu\,, & j_1^\pm \mapsto -j_1^\pm\,, & j_2^\pm \mapsto j_2^\pm\,. \end{array}$$

In particular,

$$\forall\, t \in \mathbb{R}^+ : \quad A_{R_2-\mathrm{orb}}(0, t, 0, t) = A_T\left(0, \frac{t}{2}, 0, 2t\right). \tag{L5}$$

We remark that $\mathbb{Z}_2(R)$ applied to the theory with fundamental cell such that $\tau = 1/2 + i/2, \rho = i$ has three positive and three negative eigenvalues on the set of $(1,0)$ fields. Hence it is not conjugate to any other crystallographic symmetry of $A_T(0,1,0,1)$.

5.3 Series of multicritical lines and points obtainable from (L1) and (L5)

We are now going to mod out further symmetries on both sides of the equalities obtained above. The main problem is to find the correct translation for the action of a symmetry from one model to the other. The simplest case is (L5) from which we mod out R_1 on both sides. Because all the symmetries used so far respect the factorization of $A_T(0,t,0,t)$ into a tensor product of two circle theories and commute, we directly get

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^+-\mathrm{orb}}(0, t, 0, t) = A_{R_1-\mathrm{orb}}\left(0, \frac{t}{2}, 0, 2t\right). \tag{L6}$$

Note that by mirror symmetry and T–duality (4.21) we have $A_{R_1-\mathrm{orb}}(0,2,0,1/2) = A_{R_2-\mathrm{orb}}(0,2,0,2)$, hence the above multicritical line and the one found in (L5) intersect in a tricritical point:

$$A_{D_2^+-\mathrm{orb}}(0,1,0,1) = A_{R_1-\mathrm{orb}}\left(0, \frac{1}{2}, 0, 2\right) = A_T(0,1,0,4)\,. \tag{T1}$$

We now systematically mod out all symmetries of the torus theory $A_T(0, t/2, 0, 2t)$ in (L5). The procedure is similar in all cases, namely, the charge lattices of the underlying toroidal theories on both sides of an identification must be determined, as well as twisted ground states, if present. After having performed a state by state identification, symmetries can be translated from one side to the other. This way the details which we partly omit in the proofs below can easily be filled.

As to (L5), by (3.3) the actions we can generically mod out on the torus theory $\mathcal{A}_T(0,t/2,0,2t)$ are $\mathbb{Z}_2, \mathbb{Z}_2(R), \mathbb{Z}_2(T_R), D_2^\pm, D_2(T_R)$ and $D_2(T_R')$. At $t=2$ one has additional $\widetilde{Z_2(R)}$ and $\mathbb{Z}_4$ actions which give no new identifications, though.

Modding out by $\mathbb{Z}_2(R_1)$ gives the bicritical line (L6) as discussed above. The reflection R_2 on the torus side acts as a shift by $\delta_1 = \frac{1}{2}\binom{1}{0}$ on the underlying torus theory of $\mathcal{A}_{R_2}(0,t,0,t)$ leading to a trivial identity. The symmetry T_{R_1} applied to the torus side differs in its action from R_1 by additional signs on those vertex operators (of lowest dimension) in $\mathcal{A}_T(0,t/2,0,2t)$ which correspond to twisted ground states in $\mathcal{A}_{R_2}(0,t,0,t)$. Therefore, comparison with (L6) shows

$$\forall\, t\in\mathbb{R}^+: \quad \mathcal{A}_{D_2^- -\mathrm{orb}}(0,t,0,t) = \mathcal{A}_{T_{R_1}-\mathrm{orb}}\left(0,\frac{t}{2},0,2t\right). \qquad \text{(L7)}$$

Modding out by T_{R_2} instead of T_{R_1} again gives a trivial identity, since T_{R_2} acts on the underlying torus of $\mathcal{A}_{R_2-\mathrm{orb}}(0,t,0,t)$ by the shift T_{δ_1}. Note that a comparison of (L7) with (L6) gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion. Since $\mathcal{A}_{T_{R_1}-\mathrm{orb}}(0,1/2,0,2) = \mathcal{A}_{T_{R_2}-\mathrm{orb}}(0,2,0,2)$ by T–duality (see the discussion of lattice 10), the multicritical lines (L7) and (L1) intersect in a tricritical point:

$$\mathcal{A}_{D_2^- -\mathrm{orb}}(0,1,0,1) = \mathcal{A}_{T_{R_2}-\mathrm{orb}}(0,2,0,2) = \mathcal{A}_T\left(0,1,\frac{1}{2},1\right). \qquad \text{(T2)}$$

Next, we mod out the ordinary $\mathbb{Z}_2$ action on (L5). The multicritical line (L5) can also be written as $\mathcal{A}_{\widehat{T}_{R_2}-\mathrm{orb}}(0,t,0,t) = \mathcal{A}_T(0,t/2,0,2t)$. Recall from (2.15) that $\mathcal{A}_T(0,t,0,t)$ as well as $\mathcal{A}_T(0,2t,0,t/2)$ are tensor products of circle theories at radii $r=1$, $r'=t$ and $r=2$, $r'=t$, respectively. Now consider the residual action of $D_2(T_R)$ of the original torus theory $\mathcal{A}_T(0,t,0,t)$ on the orbifoldized theory $\mathcal{A}_{\widehat{T}_{R_2}-\mathrm{orb}}(0,t,0,t)$ and note that it acts as ordinary $\mathbb{Z}_2$ on the invariant sector. The twisted ground states of the first circle factor are interchanged, so all in all we get an ordinary $\mathbb{Z}_2$ action on the torus theory $\mathcal{A}_T(0,t/2,0,2t)$. This yields

$$\forall\, t\in\mathbb{R}^+: \quad \mathcal{A}_{D_2(T_R)-\mathrm{orb}}(0,t,0,t) = \mathcal{A}_{\mathbb{Z}_2-\mathrm{orb}}\left(0,\frac{t}{2},0,2t\right). \qquad \text{(L8)}$$

By analogous arguments one finds that modding out (L1) by $\mathbb{Z}_2$ on the torus side yields

$$\forall\, t\in\mathbb{R}^+: \quad \mathcal{A}_{\mathbb{Z}_2-\mathrm{orb}}\left(0,\frac{t}{2},\frac{1}{2},\frac{t}{2}\right) = \mathcal{A}_{D_2(\mathbb{T}'_R)-\mathrm{orb}}(0,t,0,t)\,. \qquad \text{(L9)}$$

As mentioned above, R_2 applied to the torus theory $A_T(0,t/2,1/2,t/2)$ acts as shift T_{δ_1} on the underlying torus theory of $A_{R_2-\text{orb}}(0,t,0,t)$. Applying this to the bicritical line (L8), if R_2 acts with positive sign on the $\mathbb{Z}_2$ twisted ground states of the right hand side we obtain a trivial identity. On the other hand, if we use negative discrete torsion on the right hand side we find

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2(T_R)-\text{orb}}\left(0, \frac{t}{2}, 0, 2t\right) = A_{D_2^- -\text{orb}}\left(0, \frac{t}{2}, 0, 2t\right). \tag{L10}$$

Note that the bicritical lines (L7) and (L10) intersect in a tricritical point which can be interpreted as the result of modding out (T1) by T_{R_1}:

$$A_{T_{R_1}-\text{orb}}(0,1,0,4) = A_{D_2^- -\text{orb}}(0,2,0,2) = A_{D_2(T_R)-\text{orb}}\left(0, \frac{1}{2}, 0, 2\right). \tag{5.1}$$

To mod out (L5) by $D_2(T_R')$ on the torus side amounts to modding out (L7) by T_{R_2}' which acts as shift $T_{\delta'}$, $\delta' = \frac{1}{2}\binom{1}{t}$ on the underlying torus of $A_{D_2^- -\text{orb}}(0,t,0,t)$. Thus

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^- -\text{orb}}\left(\frac{1}{2}, \frac{t}{2}, 0, \frac{t}{2}\right) = A_{D_2(T_R')-\text{orb}}\left(0, \frac{t}{2}, 0, 2t\right), \tag{L11}$$

Note that because of T–duality $A_{D_2(T_R')-\text{orb}}(0,2,0,2) = A_{D_2(T_R')-\text{orb}}(0,1/2,0,2)$ as discussed for lattice 12, so (L11) intersects (L9) in a tricritical point which can be understood as the result of modding out (T2) by $\mathbb{Z}_2$:

$$A_{D_2^- -\text{orb}}\left(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}\right) = A_{D_2(T_R')-\text{orb}}\left(0, \frac{1}{2}, 0, 2\right) = A_{\mathbb{Z}_2}\left(0, 1, \frac{1}{2}, 1\right). \tag{T3}$$

We now turn to a systematic discussion of intersection lines and points obtained from (L1). From $A_T(0,t/2,1/2,t/2)$ we can generically mod out $\mathbb{Z}_2, \mathbb{Z}_2(R)$ and $D_2^\pm$. The additional symmetries for $t=1$ and $t=2$ produce nothing new.

Modding out by the ordinary $\mathbb{Z}_2$ action on the torus side gives the bicritical line (L9), as was mentioned above. We claim that the result of modding out a $\mathbb{Z}_2(R_1)$ action leads to the bicritical line

$$\forall\, t \in \mathbb{R}^+ : \quad A_{R_1-\text{orb}}\left(0, \frac{t}{2}, \frac{1}{2}, \frac{t}{2}\right) = A_{D_2(T_R)-\text{orb}}\left(0, \frac{1}{t}, 0, t\right). \tag{L12}$$

Actually, the slightly surprising parameters on the right hand side are due to an apparent asymmetry in the definition of $D_2(T_R) = \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}$. If we use $\widehat{D_2(T_R)} = \{1, A(\pi), T_{R_2}, \widehat{T}_{R_1}\}$ instead, then by T–duality (see the discussion of lattice 11) the parameters on the right hand side of (L12) are $(0,t,0,t)$. Our claim thus amounts to the fact that R_1 as applied to $A_T(0,t/2,1/2,t/2)$ induces an ordinary $\mathbb{Z}_2$ action (or equivalently $\widehat{T}_{R_1}$) on $A_{T_{R_2}-\text{orb}}(0,t,0,t)$. For the $(1,0)$ fields this is easy to check: R_1 leaves one of the abelian currents of the torus theory invariant and multiplies the other by -1. So do $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $A_{T_{R_2}-\text{orb}}(0,t,0,t)$, where the T_{R_2}

invariant generic abelian current of the underlying torus theory is multiplied by -1, and the T_{R_2} invariant combination of vertex operators remains invariant. To give a full proof for (L12), note that the charge lattice of $\mathcal{A}_T(1/2, t/2, 0, t/2)$ by (2.12) is generated by vectors

$$(p, \bar{p}) \in \left\{ \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ -1/t \end{pmatrix}, \begin{pmatrix} 1 \\ -1/t \end{pmatrix} \right), \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right), \right.$$
$$\left. \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ 2/t \end{pmatrix}, \begin{pmatrix} 0 \\ 2/t \end{pmatrix} \right), \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1/2 \\ t/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -t/2 \end{pmatrix} \right) \right\}.$$

The four vertex operators of dimension $\frac{1}{4}(1 + 1/t^2)$ given by

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi^1 + \delta\varphi^2/t)} e^{\frac{i\epsilon}{\sqrt{2}}(\bar{\varphi}^1 + \delta\bar{\varphi}^2/t)}, \qquad \epsilon, \delta \in \{\pm 1\}$$

correspond to the following T_{R_2} invariant vertex operators of $\mathcal{A}_T(0, t, 0, t)$:

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi^1_\prime + \varphi^2_\prime/t)} e^{\frac{i\epsilon}{\sqrt{2}}(\delta\bar{\varphi}^1_\prime + \bar{\varphi}^2_\prime/t)} - e^{\frac{i\epsilon}{\sqrt{2}}(-\varphi^1_\prime + \varphi^2_\prime/t)} e^{\frac{i\epsilon}{\sqrt{2}}(-\delta\bar{\varphi}^1_\prime + \bar{\varphi}^2_\prime/t)}, \qquad \epsilon, \delta \in \{\pm 1\}$$

(see (2.12) to determine the charge lattice of $\mathcal{A}_T(0, t, 0, t)$; $\varphi^\mu_\prime$ denote the bosonic fields in this torus theory to distinguish them from φ^μ on $\mathcal{A}_T(0, t/2, 1/2, t/2)$). Both R_1 on $\mathcal{A}_T(1/2, t/2, 0, t/2)$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $\mathcal{A}_{T_{R_2}-\mathrm{orb}}(0, t, 0, t)$ pairwise interchange these vertex operators. The four vertex operators of dimension $\frac{1}{16}(1 + t^2)$ given by

$$e^{\frac{i}{2\sqrt{2}}(\epsilon\varphi^1 + \delta t\varphi^2)} e^{-\frac{i}{2\sqrt{2}}(\epsilon\bar{\varphi}^1 + \delta t\bar{\varphi}^2)}, \qquad \epsilon, \delta \in \{\pm 1\}$$

correspond to the twisted ground states on $\mathcal{A}_{T_{R_2}-\mathrm{orb}}(0, t, 0, t)$, both being pairwise interchanged by R_1 on $\mathcal{A}_T(1/2, t/2, 0, t/2)$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $\mathcal{A}_{T_{R_2}-\mathrm{orb}}(0, t, 0, t)$ as well. This proves (L12). Modding out R_2 instead of R_1 gives the same result, up to T–duality. Note that the point (5.1) actually lies on (L12), hence we have found another quadrucritical point:

$$\begin{aligned} \mathcal{A}_{T_R-\mathrm{orb}}(0, 1, 0, 4) &= \mathcal{A}_{D_2^- -\mathrm{orb}}(0, 2, 0, 2) \\ &= \mathcal{A}_{D_2(T_R)-\mathrm{orb}}\left(0, \frac{1}{2}, 0, 2\right) = \mathcal{A}_{R-\mathrm{orb}}\left(0, 1, \frac{1}{2}, 1\right). \end{aligned} \qquad \text{(Q2)}$$

Moreover, (L12) intersects the bicritical lines (L2) and (L8), so there is another quadrucritical point:

$$\begin{aligned} \mathcal{A}_{R-\mathrm{orb}}\left(\frac{1}{2}, \frac{1}{2}, 0, 2\right) &= \mathcal{A}_{D_2(T_R)-\mathrm{orb}}(0, 1, 0, 1) \\ &= \mathcal{A}_{\mathbb{Z}_2-\mathrm{orb}}(0, 2, 0, 2) = \mathcal{A}_{T_{R_1}-\mathrm{orb}}(0, 2, 0, 2). \end{aligned} \qquad \text{(Q3)}$$

We proceed with the above reasoning to see that the $\mathbb{Z}_2$ action on $\mathcal{A}_T(0, t/2, 1/2, t/2)$ translates to a T'_{R_1} action on $\mathcal{A}_{T_{R_2}-\mathrm{orb}}(0, t, 0, t) = \mathcal{A}_{T'_{R_2}-\mathrm{orb}}(0, t, 0, t)$ (this is the proof of (L9)). Therefore, to determine the action induced by D_2^+ on $A_T(0,t/2,1/2,t/2)$, we note that on $A_{T_{R_2}-\mathrm{orb}}(0,t,0,t)$ the additional symmetry to mod out compared to (L12) on the underlying torus theory $A_T(0,t,0,t)$ is the combination $T'_{R_1}\widehat{T}_{R_1}$, i.e. a shift by $\delta_1 = \frac{1}{2}\binom{1}{0}$. Moreover, the $\mathbb{Z}_2$ twisted ground states in $A_{\mathbb{Z}_2-\mathrm{orb}}(0,t/2,1/2,t/2)$ are given by vertex operators which are $\widehat{T}_{R_1}$ invariant, and therefore

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^+-\mathrm{orb}}\left(0,\frac{t}{2},\frac{1}{2},\frac{t}{2}\right) = A_{D_2(T_R)-\mathrm{orb}}\left(0,\frac{2}{t},0,2t\right). \qquad \text{(L13)}$$

This can also be seen by applying R_1 to $A_{\mathbb{Z}_2-\mathrm{orb}}(0,t/2,1/2,t/2)$ in (L9). Modding out the D_2^- action on the torus side analogously gives (L11), again. Note that the bicritical line (L13) intersects (L8) and (L10), so we have found two more tricritical points:

$$A_{D_2^+-\mathrm{orb}}\left(0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = A_{D_2(T_R)-\mathrm{orb}}(0,2,0,2) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,4)\,, \qquad \text{(T4)}$$

$$A_{D_2^+-\mathrm{orb}}\left(0,1,\frac{1}{2},1\right) = A_{D_2(T_R)-\mathrm{orb}}(0,1,0,4) = A_{D_2^- -\mathrm{orb}}(0,1,0,4)\,. \qquad \text{(T5)}$$

5.4 Series of multicritical lines and points obtainable from (L2)-(L4)

To gain further identifications from (L2) we can only mod out further symmetries of the underlying torus theory $A_T(0,t,0,t)$. If we add generators of order four we only get trivial identities. An action of $\mathbb{Z}_2(R)$ type basically acts as a shift on the $A_{T_{R_1}}(0,t,0,t)$ theory, so we arrive at the bicritical lines (L6) and (L7) again. All other symmetries give trivial identities.

Next we consider (L3). The symmetries we can generically mod out are $\mathbb{Z}_2, \mathbb{Z}_2(R)$ and $\mathbb{Z}_2(T_R)$, all giving trivial identities. For $t = \sqrt{3}/2$ we can mod out additional symmetries containing a $\mathbb{Z}_3$ action, but this does not produce anything new. For the special value $t = 1/2$, where we have $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1) = A_{R_1-\mathrm{orb}}(1/2,1/2,1/2,1/2)$ all but the modding out of T'_{R_1} give trivial identities as well. The symmetry T'_{R_1} multiplies both $\mathbb{Z}_2$ invariant $(1,0)$ fields in $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$ by -1, and the generators of the invariant part of the $A_T(0,1,0,1)$ charge lattice are pairwise interchanged. The same is true for the $\mathbb{Z}_2$ twisted ground states. We claim that this translates to an R_2 action on $A_{R_1-\mathrm{orb}}(1/2,1/2,1/2,1/2)$. Namely, as a result of the discussion for lattice 7 we found that on $A_T(1/2,1/2,1/2,1/2)$ the action of D_2 leaves invariant none of the combinations of vertex operators of dimensions $(1,0)$. The respective $(1/8,1/8)$ and $(1/2,1/2)$ fields in $A_{R_1-\mathrm{orb}}(1/2,1/2,1/2,1/2)$ are also pairwise interchanged, thus

$$A_{D_2(T'_R)-\mathrm{orb}}(0,1,0,1) = A_{D_2-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

By (L9) and $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1/2,1/2,1/2) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,2)$ we see that we have actually found a tricritical point on a bicritical line:

$$A_{D_2-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,2) = A_{D_2(T'_R)-\mathrm{orb}}(0,1,0,1). \qquad \text{(T6)}$$

We remark that the above can be seen more directly by showing that in the notation of section 5.2 the groups $\widetilde{\mathbb{Z}_2(R_1)} \times \widetilde{\mathbb{Z}_2(R_2)}, \mathbb{Z}_2 \times \mathbb{Z}_2(T_{\delta'})$ and $D_2(T_R')$ are conjugate symmetry groups of type D_2 of the $\mathrm{SU}(2)^2$ torus theory.

In the discussion of lattice 15 we found that $D_4^{\pm}$ acting on $A_T(0,1,1/2,1/2)$ has a subgroup $D_2' \subset D_4^{\pm}$ which effectively acts on $A_T(1/2,1/2,1/2,1/2) = A_T(0,1,0,1)$. By the above this is conjugate to the $D_2(T_R')$ action on $A_T(0,1,0,1)$, where $D_2(T_R') \subset D_4^{\pm}(T_R')$ generically exactly gives the distinction between $D_4^{\pm}(T_R')$ and $D_4^{\pm}$. This means

$$A_{D_4^+-\mathrm{orb}}\left(0,1,\frac{1}{2},\frac{1}{2}\right) = A_{D_4(T_R')^+-\mathrm{orb}}(0,1,0,1)\,, \tag{5.2}$$

$$A_{D_4^--\mathrm{orb}}\left(0,1,\frac{1}{2},\frac{1}{2}\right) = A_{D_4(T_R')^--\mathrm{orb}}(0,1,0,1)\,. \tag{C1}$$

Let us now turn to the discussion of (L4). Generically, we can only mod out a $\mathbb{Z}_2$ action on $A_T(0,2t,1/4,t/2)$. This leads to another bicritical line:

$$\forall\, t \in \mathbb{R}^+: \quad A_{D_2-\mathrm{orb}}\left(\frac{1}{2},t,\frac{1}{2},t\right) = A_{\mathbb{Z}_2-\mathrm{orb}}\left(0,2t,\frac{1}{4},\frac{t}{2}\right), \tag{L14}$$

as follows directly from (L3) and (L4). Note that (L14) intersects the bicritical line (L9) in (T6).

We can mod out additional symmetries of (L4) at special values of t, namely if $\rho = 1/4 + it/2$ is equivalent to ρ' with $\rho_1' \in \{0,1/2\}$ by Möbius transformations. This is true for $t \in \{1/2, \sqrt{3}/2, \sqrt{7}/2, \sqrt{5/12}, \sqrt{3/20}, \sqrt{1/28}\}$, but only for $t = \sqrt{3}/2$ we produce a new identification by our methods. Here, (L4) gives $A_{R_2-\mathrm{orb}}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) = A_T(0,\sqrt{3},0,\sqrt{3})$, and the torus theory decomposes into a tensor product of two $c=1$ circle theories at radii $r=1$ and $r'=\sqrt{3}$, respectively. The latter only contains one $(1,0)$ field which is identified with the vertex operator $e^{i\sqrt{2/3}\,\varphi_1}e^{i\sqrt{2/3}\,\bar\varphi_1}+e^{-i\sqrt{2/3}\,\varphi_1}e^{-i\sqrt{2/3}\,\bar\varphi_1}$ in the $A_{R_2-\mathrm{orb}}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$ model. The SU(2) generators of the first circle factor are identified with the two other R_2 invariant vertex operators and the abelian current j_2 of $A_T(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$. The only symmetry we can mod out to find a new identification is T_{R_1}. Then by definition, of the $(1,0)$ fields on the torus side only one is invariant, namely the abelian current of the first factor theory. The same is true for the R_1 action on $A_{R_2-\mathrm{orb}}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$, where only one combination of vertex operators is invariant. Actually, the actions match entirely, showing

$$A_{D_2-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_{T_{R_1}-\mathrm{orb}}(0,\sqrt{3},0,\sqrt{3}). \tag{C2}$$

5.5 Series of multicritical points obtainable from (Q1)

The identifications in section 5.2 we have not yet used by our discussions of the bicritical lines (L1)-(L5) are $A_T(0,1,0,2) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$ and $A_{T_{R_1}-\mathrm{orb}}(0,1,0,1) = A_{R-\mathrm{orb}}(1/2,1/2,1/2,1/2)$, taken from (Q1). In the latter case we can mod out addi-

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tional symmetries on the underlying tori, but this produces no new identifications. Namely, the ordinary $\mathbb{Z}_2$ action applied to the left hand side gives the identification $A_{D_2(T_R)-\mathrm{orb}}(0,1,0,1) = A_{R_1-\mathrm{orb}}(0,1/2,1/2,1/2)$ on (L12), and $\mathbb{Z}_2$ applied to the right hand side gives $A_{D_2(T'_R)-\mathrm{orb}}(0,1,0,1) = A_{D_2-\mathrm{orb}}(1/2,1/2,1/2,1/2)$, see (T6). In fact, by the discussion at the beginning of the section we know that it suffices to mod out further symmetries of identities that contain toroidal theories.

We are now going to mod out further symmetries on both sides of the equality $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1) = A_T(0,1,0,2)$. We mostly use the description in terms of the toroidal theory $A_T(0,1,0,2)$, which by (2.12) has charge vectors

$$\overset{(-)}{p} = \frac{1}{2}\left\{ \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \pm 2 \begin{pmatrix} m_2 \\ m_1 \end{pmatrix} \right\}, \qquad m_i, n_i \in \mathbb{Z}. \tag{5.3}$$

On the $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$ side, the torus currents J_1, J_2 of $A_T(0,1,0,2)$ are $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h,\bar{h}) = (1,0)$ in the two $c=1$ factors of $A_T(0,1,0,1)$. The states $|0,0,\pm1,0\rangle$, $|0,0,0,\pm1\rangle$ in $A_T(0,1,0,2)$ by (5.3) correspond to the $(1/8,1/8)$ fields of the theory and therefore are identified with the four twisted ground states of the $\mathbb{Z}_2$ orbifold $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$. Further generators of the Hilbert space of $A_T(0,1,0,2)$ are vertex operators corresponding to $|\pm1,0,0,0\rangle$, $|0,\pm1,0,0\rangle$ which are identified with the $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h,\bar{h}) = (1/2,1/2)$ of the $A_T(0,1,0,1)$ side. These do not live in one of the separate factor theories.

The $\mathbb{Z}_2$ action on $A_T(0,1,0,2)$ induces a $\widetilde{\mathbb{Z}_2(R)}$ action on the underlying torus of $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$, and we arrive at $A_{D_2-\mathrm{orb}}(1/2,1/2,1/2,1/2) = A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,2)$ reproducing part of (T6). The R_1 action on $A_T(0,1,0,2)$ translates to $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$ in the following way: among the $(1,0)$ fields in $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$ only the combination in the first factor of $A_T(0,1,0,1)$ is invariant; two of the twisted ground states of the $\mathbb{Z}_2$ orbifold are exchanged, whereas two of them are fixed. Among the $(1/2,1/2)$ fields, again two are fixed and two are exchanged; this is just the R_1 action on $A_{\mathbb{Z}_2-\mathrm{orb}}(1/2,1/2,0,1)$, hence

$$A_{D_2^+-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,1\right) = A_{R-\mathrm{orb}}(0,1,0,2)\,. \tag{C3}$$

If we combine the $\mathbb{Z}_2$ and $\mathbb{Z}_2(R)$ actions on $A_T(0,1,0,2)$, the $\mathbb{Z}_2$ now will act as a shift on the underlying torus of $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1)$. It is easier to understand the resulting identification by considering the $\mathbb{Z}_2$ orbifold theory $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,2)$. T'_{R_1} acts on $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,2)$ by pairwise interchanging the $\mathbb{Z}_2$ twisted ground states and multiplying the $\mathbb{Z}_2$ invariant vertex operators of dimensions $(1/8,1/8)$ in $A_T(0,1,0,2)$ by -1. On the other hand, R_1 with negative discrete torsion will multiply the two T'_{R_1} invariant twisted ground state combinations by -1 but leave invariant the two $\mathbb{Z}_2$ invariant $(1/8,1/8)$ fields of $A_T(0,1,0,2)$. These $\mathbb{Z}_2$ actions are conjugate, since the action on the invariant $\mathbb{Z}_2$ twisted ground state combinations of $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1) =$

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$A_T(0,1,0,2)$ is merely exchanged with that on two combinations of twisted ground states of $A_{\mathbb{Z}_2-\text{orb}}(0,1,0,2)$. This again is possible because of the $c=1$ identification between the circle theory at radius $r=1$ and the orbifold theory at radius $r=2$. In summary,

$$A_{D_2(T_R')-\text{orb}}(0,1,0,2) = A_{D_2^- -\text{orb}}(0,1,0,2)\,. \tag{C4}$$

The T_{R_1} action on $A_T(0,1,0,2)$ differs from the R_1 action by a sign in the action on the $(1/8,1/8)$ fields, i.e. the twisted ground states of the $\mathbb{Z}_2$ orbifold on the $A_{\mathbb{Z}_2-\text{orb}}(1/2,1/2,0,1)$ side. Therefore by comparison with (C3)

$$A_{D_2^- -\text{orb}}\left(\frac{1}{2},\frac{1}{2},0,1\right) = A_{T_R-\text{orb}}(0,1,0,2)\,. \tag{C5}$$

Comparison of (C3) with (C5) also gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion.

If we mod out $\widetilde{\mathbb{Z}_2(R)}$ and the corresponding D_2 type symmetries on $A_T(0,1,0,2)$, i.e. consider $\mathbb{Z}_2(R)$ on $A_T(1/2,1/2,0,2)$ we only reproduce identities we have found already above: $A_{D_2(T_R)-\text{orb}}(0,1,0,1) = A_{R-\text{orb}}(1/2,1/2,0,2)$ on (L12), as well as $A_{D_2(T_R^{(\prime)})-\text{orb}}(0,2,0,2) = A_{D_2^{\pm}-\text{orb}}(1/2,1/2,0,2)$ on (L13) and (L11), respectively.

Next we discuss the action of T_{R_1} on $A_T(0,1,0,1/2)$ instead of $A_T(0,1,0,2)$. In (5.3) this exchanges the roles of m_i and n_i, such that compared to the action of R_1 on $A_T(0,1,0,2)$ we now have additional signs on $(1/2,1/2)$ fields. In particular, only one combination of $(1/2,1/2)$ fields is invariant, as well as three of the twisted ground state combinations in $A_{\mathbb{Z}_2-\text{orb}}(0,1,0,1)$. We claim that this is the residual action of an ordinary $\mathbb{Z}_4$ rotation on $A_T(0,1,0,1)$. It acts by interchanging the two circle factors of $A_T(0,1,0,1)$, but the generators of the Hilbert space of the second factor are multiplied with an additional sign. Indeed, this is exactly the T_{R_1} action on a torus whose lattice has an additional generator $(1/2,1/2)$ compared to $\mathbb{Z}^2$ for $A_T(0,1,0,1)$, i.e. on $A_T(0,1,0,1/2)$. Hence,

$$A_{\mathbb{Z}_4-\text{orb}}(0,1,0,1) = A_{T_R-\text{orb}}\left(0,1,0,\frac{1}{2}\right). \tag{C6}$$

Using (C6) we can further mod out T_{R_2} on the underlying torus theory of the above $A_{T_{R_1}-\text{orb}}(0,1,0,1/2)$. This translates to a $\widetilde{\mathbb{Z}_2(R_2)}$ action on the underlying torus theory of $A_{\mathbb{Z}_4-\text{orb}}(0,1,0,1)$, so

$$A_{D_4^+-\text{orb}}\left(0,1,\frac{1}{2},\frac{1}{2}\right) = A_{D_2(T_R)-\text{orb}}\left(0,1,0,\frac{1}{2}\right).$$

By (5.2) we see that we have actually found a tricritical point:

$$A_{D_4^+-\text{orb}}\left(0,1,\frac{1}{2},\frac{1}{2}\right) = A_{D_2(T_R)-\text{orb}}\left(0,1,0,\frac{1}{2}\right) = A_{D_4(T_R')^+-\text{orb}}(0,1,0,1)\,. \tag{T7}$$

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We now rewrite (C6) as $A_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,1) = A_{T'_{R_1}-\mathrm{orb}}(0,1,0,1/2)$ and mod out by T'_{R_2} on the underlying torus of the right hand side. Analogously to the T_{R_2} action on $A_{T_{R_1}-\mathrm{orb}}(0,t/2,0,2t)$ in (L7), which induced a shift on the underlying torus theory of $A_{D_2^- -\mathrm{orb}}(0,t,0,t)$, in (C6) we get a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $A_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,1)$. Then we obtain

$$A_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,2) = A_{D_2(T'_R)-\mathrm{orb}}\left(0,1,0,\frac{1}{2}\right). \tag{C7}$$

Back to the identification $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,1) = A_T(0,1,0,1/2)$ in (Q1) we now mod out groups containing $\mathbb{Z}_4$ on the torus side. With the ordinary $\mathbb{Z}_4$ action we reproduce the above bicritical point (C7), but in combination with $D_2(T'_R)$, the $\mathbb{Z}_4$ generator acts as a shift on the underlying torus theory of $A_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,2)$ in (C7):

$$A_{D_4(T'_R)^+-\mathrm{orb}}\left(0,1,0,\frac{1}{2}\right) = A_{\mathbb{Z}_4-\mathrm{orb}}(0,1,0,4) \tag{C8}$$

$$A_{D_4(T'_R)^--\mathrm{orb}}\left(0,1,0,\frac{1}{2}\right) = A_{\mathbb{Z}_4-\mathrm{orb}}\left(0,1,\frac{1}{2},1\right). \tag{C9}$$

The latter identification is more easily understood when we mod out symmetries on the tricritical point (T3), as we will do in section 5.7.

The effect of D_4 type actions is most easily understood from the fact that by (C4) the action of $D_2(T'_R) \subset D_4(T'_R)^{\pm}$ on $A_T(0,1,0,2)$ is conjugate to that of $D_2^- \subset D_4^{-\pm}$. Therefore,

$$A_{D_4(T'_R)^+-\mathrm{orb}}(0,1,0,2) = A_{D_4^{-+}-\mathrm{orb}}(0,1,0,2) \tag{5.4}$$

$$A_{D_4(T'_R)^--\mathrm{orb}}(0,1,0,2) = A_{D_4^{--}-\mathrm{orb}}(0,1,0,2)\,. \tag{5.5}$$

5.6 Series of multicritical points obtainable from (T1)

From the multicritical points and lines determined so far we can find further multicritical points by modding out further symmetries. By the systematic procedure we followed above, this can only give something new, if we use an identification obtained as intersection of bicritical lines. Moreover, because by the discussion at the beginning of the section it suffices to use identifications containing a toroidal theory, only (T1) and (T2) are left to be discussed in this and the following section.

For the point (T1) only the identification $A_T(0,1,0,4) = A_{D_2^+}(0,1,0,1)$ has not been used yet. By modding out $\mathbb{Z}_2$ we yield (T4) from (T1), in particular $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,0,4) = A_{D_2^+-\mathrm{orb}}(0,1/2,1/2,1/2)$. Modding out a $\mathbb{Z}_2(R)$ action yields $A_{R-\mathrm{orb}}(0,1,0,4) = A_{D_2^+-\mathrm{orb}}(0,2,0,2)$ on (L6). Note that this shows that $\mathbb{Z}_2$ and R on $A_T(0,1,0,4)$ both induce shifts on the underlying torus theory of $A_{D_2^+}(0,1,0,1)$, namely $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$, and $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, respectively. The combined action gives a trivial identity for D_2^+, and $A_{D_2^--\mathrm{orb}}(0,1,0,4) = A_{D_2^+-\mathrm{orb}}(0,1,1/2,1)$ in (T5). Modding out $\mathbb{Z}_2(T_R), D_2(T_R)$ and $D_2(T'_R)$ reproduces the points at $t=2$ in (L7), (L10), and (L11), respectively. Modding out $\mathbb{Z}_4$ reproduces (C8). To determine the result of modding out D_4 actions, note that by the above the action of R induces a shift T_{δ_1} on the underlying torus theory of $A_{D_2^+}(0,1,0,1)$, so from (C8) we obtain

$$A_{D_4^{--}-\mathrm{orb}}(0,1,0,4) = A_{D_4(T'_R)^+-\mathrm{orb}}(0,1,0,4)\,. \tag{5.6}$$

All the other choices of discrete torsion give trivial identities. Modding out by $D_4(T'_R)^\pm$ gives the same or a trivial identity again.

Next we mod out $\widetilde{\mathbb{Z}_2(R)}$, i.e. $\mathbb{Z}_2(R)$ on $A_T(1/2,1/2,0,4)$. This interchanges the two circle factors of the original $A_T(0,1,0,1)$ in $A_{D_2^+-\mathrm{orb}}(0,1,0,1)$ above and thus is equivalent to adding a $\mathbb{Z}_4$ generator to D_2. Therefore,

$$A_{R-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,4\right) = A_{D_4^{++}-\mathrm{orb}}(0,1,0,1)\,. \tag{C10}$$

To mod out the corresponding D_2 actions we again use the above observation that $\mathbb{Z}_2$ on $A_T(1/2,1/2,0,4)$ acts as T_{δ_1} on the underlying torus theory of $A_{D_2^+-\mathrm{orb}}(0,1,0,1)$ to find

$$A_{D_2^+-\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,4\right) = A_{D_4^{++}-\mathrm{orb}}(0,1,0,2)\,, \tag{C11}$$

and

$$A_{D_2^--\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,4\right) = A_{D_4(T'_R)^+-\mathrm{orb}}(0,1,0,2)\,,$$

where the latter together with (5.4) gives a tricritical point

$$A_{D_2^--\mathrm{orb}}\left(\frac{1}{2},\frac{1}{2},0,4\right) = A_{D_4(T'_R)^+-\mathrm{orb}}(0,1,0,2) = A_{D_4^{-+}-\mathrm{orb}}(0,1,0,2)\,. \tag{T8}$$

5.7 Series of multicritical points obtainable from (T2)

We now discuss additional identifications that can be obtained from (T2). The only identity not used up to now is $A_T(0,1,1/2,1) = A_{D_2^--\mathrm{orb}}(0,1,0,1)$. If we mod out a $\mathbb{Z}_2$ action from the torus theory, (T2) is transformed into (T3), in particular we yield $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,1/2,1) = A_{D_2^--\mathrm{orb}}(1/2,1/2,0,1/2)$. The $\mathbb{Z}_2$ action thus induces a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $A_{D_2^--\mathrm{orb}}(0,1,0,1)$. The R action on $A_T(0,1,1/2,1)$ induces a shift as well, now by $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, yielding $A_{R-\mathrm{orb}}(0,1,1/2,1) = A_{D_2^--\mathrm{orb}}(0,2,0,2)$ in (Q2). The combined R and $\mathbb{Z}_2$ actions thus yield a trivial identity for D_2^- and $A_{D_2^+-\mathrm{orb}}(0,1,1/2,1) = A_{D_2(T_R)-\mathrm{orb}}(0,1,0,4)$ on (L13). Modding out $\mathbb{Z}_4$ is equivalent to modding out another $\mathbb{Z}_2$ action on $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,1/2,1) = A_{D_2^--\mathrm{orb}}(1/2,1/2,0,1/2)$ which interchanges the circle factors of the underlying geometric torus (i.e. $\mathbb{Z}_2$ invariant vertex operators with $h = \bar{h}$). The action matches a $\widetilde{D_4}$ action on $A_{D_2^--\mathrm{orb}}(1/2,1/2,0,1/2)$, where the additional D_2^- invariant vertex operators as compared to $A_{D_2^--\mathrm{orb}}(1/2,1/2,0,1)$ correspond to the $\mathbb{Z}_2$ twisted ground states of $A_{\mathbb{Z}_2-\mathrm{orb}}(0,1,1/2,1)$. We thus obtain

$A_{\mathbb{Z}_4-\text{orb}}(0,1,1/2,1) = A_{D_4(T'_R)^- - \text{orb}}(0,1,0,1/2)$ reproducing (C9). Since by the above we know that R_1 on $A_T(0,1,1/2,1)$ induces a T_{δ_1} shift on the underlying torus theory of $A_{D_2^- - \text{orb}}(0,1,0,1)$, it also follows that

$$A_{D_4^- - \text{orb}}\left(0,1,\frac{1}{2},1\right) = A_{D_4(T'_R)^- - \text{orb}}(0,1,0,4)\,. \tag{C12}$$

Flipping the sign of discrete torsion on both sides of the above equivalence we find

$$A_{D_4^+ - \text{orb}}\left(0,1,\frac{1}{2},1\right) = A_{D_4(T'_R)^+ - \text{orb}}(0,1,0,4)\,,$$

which together with (5.6) yields a tricritical point:

$$A_{D_4^+ - \text{orb}}\left(0,1,\frac{1}{2},1\right) = A_{D_4(T'_R)^+ - \text{orb}}(0,1,0,4) = A_{D_4^{--} - \text{orb}}(0,1,0,4)\,. \tag{T9}$$

We now mod out $\widetilde{\mathbb{Z}_2(R)}$ on $A_T(0,1,1/2,1)$, i.e. $\mathbb{Z}_2(R)$ on $A_T(1/2,1/2,1/2,1)$. Similarly to (C10) we find

$$A_{R-\text{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right) = A_{D_4^{--} - \text{orb}}(0,1,0,1)\,. \tag{C13}$$

Because by the above, $\mathbb{Z}_2$ on $A_T(1/2,1/2,1/2,1)$ induces a shift $T_{\delta'}$ on the underlying torus theory of $A_{D_4^{--} - \text{orb}}(0,1,0,1)$ in (C13), we find

$$A_{D_2-\text{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right) = A_{D_4(T'_R)^- - \text{orb}}(0,1,0,2)\,.$$

Together with (5.5) this gives another tricritical point:

$$A_{D_2-\text{orb}}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1\right) = A_{D_4(T'_R)^- - \text{orb}}(0,1,0,2) = A_{D_4^{--} - \text{orb}}(0,1,0,2)\,. \tag{T10}$$

5.8 Multicritical points obtained from conjugate $\mathbb{Z}_3, D_3, \mathbb{Z}_6$ and D_6 type actions

We start by comparing all $\mathbb{Z}_3$ type symmetries of the SU(3) torus theory at parameters $\tau = \rho = \omega$, $\omega := e^{2\pi i/3}$. The generically conserved currents of the torus theory we call j_1, j_2, and k_1, k_2, k_3 together with $l_\mu = k_\mu^\dagger, \mu \in \{1,2,3\}$ denote the additional vertex operators with dimensions $(h,\bar{h}) = (1,0)$. The fields j_μ, k_μ, l_μ generate an $\mathrm{SU}(3)_1$ Kac–Moody algebra, and $\{k_\mu\}$, $\{l_\mu\}$ form closed orbits under the ordinary $\mathbb{Z}_3$ action. In passing we remark that among all possible $\mathbb{Z}_2$ symmetries of $A_T(\omega,\omega)$, those conjugate only reproduce (L3).

Among the $\mathbb{Z}_3$ actions on one hand we have the ordinary rotational $\mathbb{Z}_3$ which leaves two fields $k_1 + k_2 + k_3$ and $l_1 + l_2 + l_3$ invariant, three fields $j^+ = j_1 + ij_2, k_1 +$

$\omega k_2 + \omega^2 k_3, l_1 + \omega l_2 + \omega^2 l_3$ have eigenvalue ω. On the other hand, the shift orbifold by $\delta = \frac{1}{2}(\lambda_1 - \lambda_2)$ exhibits the same spectrum, where the λ_i as usual denote a basis of the lattice associated to the parameters $\tau = \rho = \omega$. Here, j_1, j_2 are invariant, and k_1, k_2, k_3 have eigenvalue ω. We particularly see that the two $\mathbb{Z}_3$ actions are conjugate, thus modding out $A_T(\omega,\omega)$ by these two symmetries gives isomorphic theories. The shift orbifold again produces a torus theory with same parameter $\tau = \omega$, but ρ reduced by a factor of three; in the following we use $\alpha := 1/2 + i3\sqrt{3}/2$ which is related to $\omega/3$ by the Möbius transformation $\mathbb{T}^2 S$ and state

$$A_{\mathbb{Z}_3-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_T\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}\right). \tag{C14}$$

We will now mod out additional symmetries on both sides of the above equality. Only those of order two give new identifications. Note that both R_2 and the ordinary $\mathbb{Z}_2$ on $A_T(\omega,\omega)$ interchange the two $\mathbb{Z}_3$–invariant $(1,0)$ fields $k_1+k_2+k_3$ and $l_1+l_2+l_3$. Thus $R_2, \mathbb{Z}_2$ must act as R_1, R_2 on the torus theory $A_T(\omega,\alpha)$. Study the action on the charge lattice to check that the order above is indeed correct. This means that the R_1 action on $A_T(\omega,\omega)$ must induce the ordinary $\mathbb{Z}_2$ action on $A_T(\omega,\alpha)$. In particular, the fields $k_1+k_2+k_3$ and $l_1+l_2+l_3$ are multiplied by -1 under R_1. Here we can confirm our result of the discussion of lattice 7: The signs obtained there occur in a completely natural way in the present example.

All in all for the $\mathbb{Z}_2$ actions on $A_T(\omega,\omega)$ compared to $A_T(\omega,\alpha)$ we have found $(R_1, R_2, \mathbb{Z}_2) \mapsto (R_2, \mathbb{Z}_2, R_1)$ and therefore directly obtain the following bicritical points:

$$A_{D_3(R_1)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_{\mathbb{Z}_2-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}\right), \tag{C15}$$

$$A_{D_3(R_2)-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_{R_1-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}\right), \tag{C16}$$

$$A_{\mathbb{Z}_6-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_{R_2-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}\right), \tag{C17}$$

$$A_{D_6-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}\right) = A_{D_2-\mathrm{orb}}\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}\right). \tag{C18}$$

6. Product theories within the moduli space

If our description of nonisolated components of $\mathcal{C}^2$ is complete, it must be possible to find all nonisolated components known so far. In particular, we should consider tensor products of known models. The simplest case is the product of two models with central charge $c=1$. The possible factor theories then are $A^{c=1}(r)$, $A^{c=1}_{\mathrm{orb}}(r)$, $A^{c=1}_T$, $A^{c=1}_O$, and $A_I^{c=1}$, corresponding to compactification on a circle with radius r, its $\mathbb{Z}_2$ orbifold or one of the three isolated components of the $c = 1$ moduli space, respectively. Models containing one of the latter three factor theories are exceptional but of course easily constructed, as was mentioned in section 2. Moreover,

$$A^{c=1}(r) \otimes A^{c=1}(r') = A_T\left(0, \frac{r'}{r}, 0, rr'\right),$$
$$A^{c=1}(r) \otimes A^{c=1}_{\text{orb}}(r') = A_{R_1-\text{orb}}\left(0, \frac{r'}{r}, 0, rr'\right),$$

and

$$A^{c=1}_{\text{orb}}(r) \otimes A^{c=1}_{\text{orb}}(r') = A_{D_2^+-\text{orb}}\left(0, \frac{r'}{r}, 0, rr'\right)$$

are obvious (see (2.15), (4.27), (4.37)).

Using the results of [6], nonisolated components of the moduli space can also be obtained by tensoring $N = 1$ superconformal field theories $A_\bullet^{c=3/2}(r)$ with central charge $c = 3/2$ with the unique unitary conformal field theory at $c = 1/2$. In this section we discuss how the resulting models $A_\bullet^M(r)$ can be found within the components of $\mathcal{C}^2$ we have determined in section 4.

By [6], the moduli space of $N = 1$ superconformal field theories with $c = 3/2$ contains five connected lines. The *circle line* $A^{c=3/2}_{\text{circ}}(r)$ is obtained from the $c = 1$ circle theories by adding one Majorana fermion, i.e. tensoring with the unique unitary conformal field theory at $c = 1/2$, the Ising model. Since the tensor product of two Ising models has a bosonic description as $\mathbb{Z}_2$ orbifold of the $c = 1$ circle theory at radius $r' = \sqrt{2}$, by the discussion of lattice 6 we directly obtain

$$A^M_{\text{circ}}(\sqrt{2}r) = A^{c=1}_{\text{orb}}(\sqrt{2}) \otimes A^{c=1}(\sqrt{2}r) = A_{R_2-\text{orb}}(0, r, 0, 2r).$$

The other four lines in the $c = 3/2$ moduli space are obtained as orbifold models of $A^{c=3/2}_{\text{circ}}(r)$. The ordinary $\mathbb{Z}_2$ orbifold generates the so-called *orbifold line* $A^{c=3/2}_{\text{orb}}(r)$. For the fermions the orbifold procedure effectively only exchanges boundary conditions, which we forget about in our $c = 2$ purely bosonic language. Therefore, we can regard $\mathbb{Z}_2$ as only acting on the second circle factor of $A^M_{\text{circ}}(\sqrt{2}r) = A_{R_2-\text{orb}}(0, r, 0, 2r)$. This amounts to modding out an R_1 action, i.e.

$$A^M_{\text{orb}}(\sqrt{2}r) = A_{D_2^+-\text{orb}}(0, r, 0, 2r).$$

Note that by the results of section 5 and in agreement with [6] the only intersection point of the above lines is situated on (L6):

$$A^M_{\text{circ}}(2) = A^M_{\text{orb}}(1).$$

The *superaffine line* $A^{c=3/2}_{s-a}(r)$ is the orbifold of $A^{c=3/2}_{\text{circ}}(r)$ by the $\mathbb{Z}_2$ type group generated by $S_\delta := t_\delta\,(-1)^{F_S}$. Here, $t_\delta = e^{2\pi i p \frac{\delta}{\sqrt{2}}}$ is the shift orbifold on the bosonic

$c = 1$ theory, and $(-1)^{F_S}$ is the spacetime fermion number operator. $(-1)^{F_S}$ acts by multiplication with -1 on the Ramond sector and trivially on the Neveu–Schwarz sector of the theory. To determine $A^M_{s-a}(\sqrt{2}r)$, we trivially continue the action of S_δ to $A^M_{\text{circ}}(\sqrt{2}r)$. Then S_δ remains to act as ordinary shift orbifold on the second factor theory in $A^M_{\text{circ}}(\sqrt{2}r)$, the $c = 1$ circle theory at radius $\sqrt{2}r$. On the first factor, we have the action of $(-1)^{F_S}$ on one of the Majorana fermions. We use the bosonic description as $\mathbb{Z}_2$ orbifold of the $c = 1$ circle theory at radius $\sqrt{2}$. Here, the Ramond sector is built on those Hilbert space ground states with odd label of the momentum mode. Thus on the underlying $c = 1$ circle theory, $(-1)^{F_S}$ acts as shift orbifold as well. This means that $A^M_{s-a}(\sqrt{2}r)$ can be obtained as shift orbifold by $T_{\delta'}$, $\delta' = \frac{1}{\sqrt{2}}\binom{1}{r}$ on the underlying torus theory $A_T(0, r, 0, 2r)$ of $A^M_{\text{circ}}(\sqrt{2}r)$:

$$A^M_{s-a}(\sqrt{2}r) = A_{R_2-\text{orb}}\left(\frac{1}{2}, \frac{r}{2}, 0, r\right).$$

The *superorbifold line* $A^{c=3/2}_{s-\text{orb}}(r)$ is a D_2 type orbifold of $A^{c=3/2}_{\text{circ}}(r)$ by the group generated by the ordinary $\mathbb{Z}_2$ action and S_δ. Since by the above $\mathbb{Z}_2$ and S_δ act as reflection R_1 and shift $T_{\delta'}$ on the underlying torus theory $A_T(0, r, 0, 2r)$ of $A^M_{\text{circ}}(\sqrt{2}r)$, respectively, we find

$$A^M_{s-\text{orb}}(\sqrt{2}r) = A_{D_2^+-\text{orb}}\left(\frac{1}{2}, \frac{r}{2}, 0, r\right).$$

By the results of section 5 we see that only the superorbifold line intersects one of the other three lines discussed so far, namely in (C3):

$$A^M_{s-\text{orb}}(\sqrt{2}) = A^M_{\text{circ}}(\sqrt{2}).$$

This agrees with the results of [6]. Finally, the *orbifold-prime line* $A^{c=3/2}_{\text{orb}'}(r)$ is obtained by modding out $S_R := (-1)^{F_S} \cdot (-1)$ from $A^{c=3/2}_{\text{circ}}(r)$, where (-1) is the generator of the ordinary $\mathbb{Z}_2$ action. In particular, for the partition functions of orbifold and orbifold–prime theories, one has the relation

$$Z^{c=3/2}_{\text{orb}'}(r) = Z^{c=3/2}_{\text{orb}}(r) - 3. \tag{6.1}$$

Since the generator of the ordinary $\mathbb{Z}_2$ action on $A^M_{\text{circ}}(\sqrt{2}r)$ acts as reflection R_1, and $(-1)^{F_S}$ is the shift orbifold on the underlying $c = 1$ circle theory at radius $\sqrt{2}$ of the first factor in $A^M_{\text{circ}}(\sqrt{2})$, S_R acts as T_{R_1} on the underlying torus theory $A_T(0, r, 0, 2r)$ of $A^M_{\text{circ}}(\sqrt{2}r)$. Therefore,

$$A^M_{\text{orb}'}(\sqrt{2}r) = A_{D_2(T_R)-\text{orb}}(0, r, 0, 2r).$$

Concerning intersections of the orbifold–prime line with the other lines discussed above, again we are in exact agreement with the results of [6]: we find multicritical points on (L13) and (L12), namely

$$A^M_{\text{orb}'}(2) = A^M_{s-\text{orb}}(2), \qquad A^M_{\text{orb}'}(1) = A^M_{s-a}(2).$$

It is a straightforward calculation to check (6.1) for our $c = 2$ models, i.e.

$$Z_{D_2(T_R)-\mathrm{orb}}(0, r, 0, 2r) = Z_{D_2^{+}-\mathrm{orb}}(0, r, 0, 2r) - 3Z_{\mathrm{Ising}}$$

from (4.37), (4.42), and (4.36).

The above in particular gives a geometric interpretation in terms of crystallographic orbifolds to all the nonisolated orbifolds discussed in [6].

7. Conclusions

We have explicitly constructed the parameter spaces and the one loop partition functions of the sixteen types of crystallographic orbifold conformal field theories of toroidal theories with central charge $c = 2$. Taking into acount all possible choices of the B–field and all values of discrete torsion, this yields 28 different components of the moduli space $\mathcal{C}^2$ of unitary conformal field theories with central charge $c = 2$. We have argued that this way, apart from the exceptional cases related to the binary tetrahedral, octahedral and icosahedral subgroups of SU(2), we get all the nonisolated irreducible components of the moduli space that can be obtained by an orbifold procedure. In the construction of the various theories some unexpected effects of the B–field have occured which might lead to a better understanding of its properties, also for higher dimensional cases.

We have determined all the multicritical points and lines of the 28 components of $\mathcal{C}^2$ constructed before. We have found fourteen bicritical lines and 31 multicritical points, among them three quadrucritical and ten tricritical points. We have proven multicriticality on the level of the operator algebra for all these lines and points. The case by case study also sheds some light on the effect of discrete torsion.

Drawing a picture of the moduli space $\mathcal{C}^2$ one will notice a complicated graph like structure with a lot of loops. In particular, by our analysis of multicritical points, all but four of the irreducible crystallographic components of the moduli space are directly or indirectly connected to the moduli space of toroidal theories. The remaining four components are $\mathcal{C}^{(0)}_{D_4^{+-}-\mathrm{orb}}, \mathcal{C}^{(0)}_{D_6^{\pm}-\mathrm{orb}}, \mathcal{C}^{(0)}_{D_3(R)-\mathrm{orb}}$.

We have related our results to those on $c = 3/2$ superconformal field theories [6]. This was done by determining the tensor products of the five continuous lines of $c = 3/2$ superconformal field theories discussed in [6] with an Ising model in terms of our description of $\mathcal{C}^2$. All multicritical points in the $c = 3/2$ moduli space are reidentified by our results on $\mathcal{C}^2$. In particular, this gives geometric interpretations to all nonisolated orbifolds discussed in [6] in terms of crystallographic orbifolds.

A discussion of the exceptional components of $\mathcal{C}^2$ is not carried out in this work. By our results, these would yield the only possible examples of asymmetric orbifold conformal field theories [18] with $c = 2$ and therefore should be studied separately. Neither do we touch the determination of isolated components of the moduli space, which is expected to be even more involved. Apart from that, our results do not give a complete classification of unitary conformal field theories with central charge $c = 2$, since we are lacking a theorem which would tell us that all nonisolated components of the moduli space may be obtained by some orbifold procedure from a subspace of the toroidal component. It would also be interesting to determine those theories in $\mathcal{C}^2$ which admit supersymmetry.

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Erratum

Erratum concerning section 4.10, page 26

It is well known [23], that the degrees of freedom due to discrete torsion in a G-orbifold are parametrized by the group cohomology $H^2(G, U(1))$. The latter is trivial for cyclic groups $G = \mathbb{Z}_M$ and for the dihedral groups $G = D_{2M+1}$, whereas $H^2(D_{2M}, \mathrm{U}(1)) = \mathbb{Z}_2$ (see [1]), or our explicit calculation at
`http://www.physics.unc.edu/~wendland/jhep/erratum.html`).

This is in agreement with our results for D_2, D_3, and D_6 orbifolds. In the first part of formula (4.46) all terms $D_4^{\delta\varepsilon}, D_4^{\delta}, D_4^{\varepsilon}$ are to be replace by $D_4^{\pm}$, since otherwise the action of D_4 on the twisted sectors is not well defined.

Hence D_4 orbifolds form only *four* families $\mathcal{C}^{(\rho)}_{D_4^{\pm}}$, $\rho \in \{0, 1/2\}$. The number of nontoroidal components of the moduli space $\mathcal{C}^2$ of unitary conformal field theories with $c = 2$ is *26*. *Three* of them are not connected to the moduli space $\mathcal{T}^2$ of toroidal theories in $\mathcal{C}^2$, namely $\mathcal{C}^{(0)}_{D_6^{\pm}-orb}$ and $\mathcal{C}^{(0)}_{D_3(R)-orb}$. The multicritical point (T8) (see page 40) hence *is only bicritical.*

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